MATEMATIKA 2

seminari

studij: **Prehrambena tehnologija**

i Biotehnologija

Sadržaj

I Integralni račun

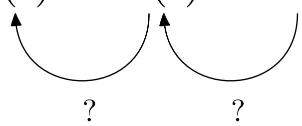
1 Integralni račun funkcije jedne varijable

1.1 Uvod

Razmotrimo odmah na početku pitanje čemu služi integral i gdje se upotrebljava:

1. Mjerni problemi kao što su: izračunavanje površine, duljine luka (opseg), volumena, oplošja. Primjeri iz fizike:

$$s(t) \overset{\frac{d}{dt}}{\mapsto} v(t) \overset{\frac{d}{dt}}{\mapsto} a(t)$$



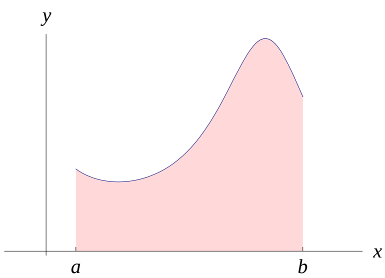
2. Rješavanje diferencijalnih jednadžbi: Znamo iz f izračunati f', a sad je pitanje kako iz f' doći do f, tj. potrebno je odrediti $y(x)$ koji zadovoljava $y'(x) = f(x)$.

1.2 Određeni (Riemannov[1]) integral. Problem površine.

U osnovnoj i srednjoj školi naučili smo kako izračunati površinu pravokutnika, trokuta, kružnice itd. Sada se postavlja pitanje kako odrediti površinu likova koji nisu tako "pravilni" kao npr. ovaj:

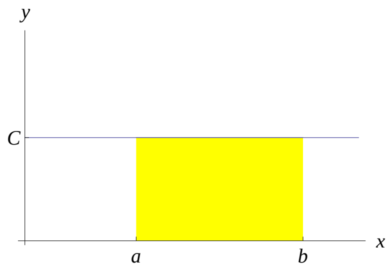
Kako bi razvili svoju intuiciju i razmišljanje promotrimo već poznate primjere na način koji će nam biti koristan pri razmišljanju o površini likova kao na prethodnoj slici.

[1] Georg Friedrich Bernhard Riemann(1826-1866), slavni njemački matematičar



1. Površina:

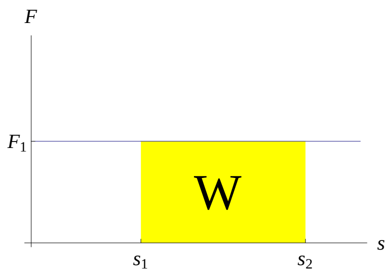
$$x \in [a, b] \quad , \quad f(x) = C$$



Od prije nam je već poznato da je površina ovog lika $P = C(b - a)$

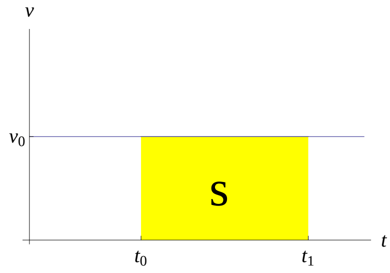
2. Rad:

$$s \in [s_1, s_2] \quad , \quad F(s) = F_1$$



Znamo da je rad jednak umnošku sile i puta tj. $W = F_1(s_2 - s_1)$

3. Put koji smo prešli u vremenskom periodu $t_1 - t_0$ brzinom v_0 jednak je $s = v_0(t_1 - t_0)$



Pretpostavimo da imamo ograničenu nenegativnu funkciju $f : [a, b] \to \mathbb{R}$ te pogledajmo skup $\Omega = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$. Skup Ω zovemo još i **krivocrtni trapez** ili **pseudotrapez**. Želimo tom skupu Ω izračunati površinu.

Ideja: Metoda iscrpljivanja
Podijelit ćemo zadani interval $[a, b]$ na manje intervale. Tu podijelu intervala $[a, b]$ nazivamo **subdivizijom** i označiti ćemo ju sa D. Dakle, imamo sljedeće:

$$D \ldots a = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_n = b$$

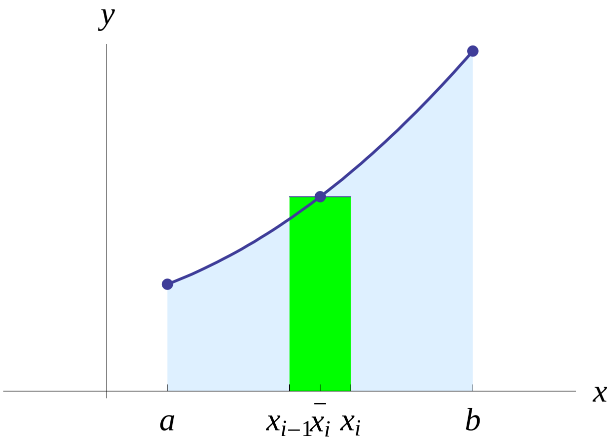
Nakon podijele intervala $[a, b]$ na manje intervale $[x_{i-1}, x_i], i = 1, \ldots, n$, iz svakog od podintervala izaberemo međutočke $\overline{x}_i \in [x_{i-1}, x_i]$.
Sada pomoću izabranih podintervala i međutočaka definiramo **intergralnu ili Riemannovu sumu** koju ćemo označiti sa $S(D)$ na sljedeći način:

$$S(D) = \sum_{i=1}^{n} f(\overline{x}_i)(x_i - x_{i-1})$$

Sljedeća slika ilustrira što smo napravili na razini samo jednog podintervala subdivizije D.

Prije prve definicije potrebno je uvesti pojam **očice subdivizije** koju ćemo označavati sa $m(D)$ i definirati na sljedeći način:

$$m(D) = \max\{x_i - x_{i-1} : i = 1, \ldots, n\}$$



Intuitivno, **očica subdivizije** je širina najvećeg podintervala u izabranoj subdiviziji D.

Definicija 1 *Kažemo da je funkcija $f : [a, b] \to \mathbb{R}$ integrabilna ako postoji*

$$\lim_{m(D)\to 0} S(D) = \lim_{m(D)\to 0} \sum_{i=1}^{n} f(\overline{x}_i)(x_i - x_{i-1})$$

Navedeni limes (ako postoji !) označavamo s:

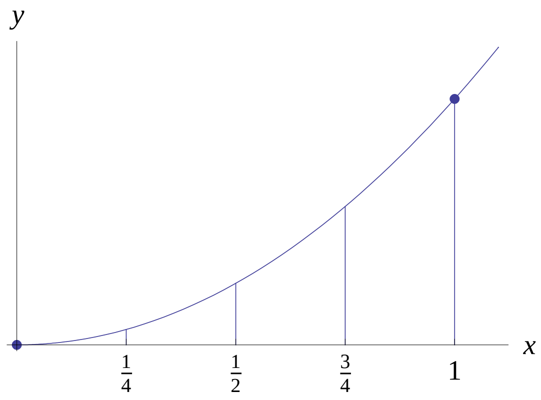
$$\int_a^b f(x)dx = \lim_{m(D)\to 0} S(D)$$

i nazivamo ***određeni integral*** *funkcije f na intervalu $[a, b]$. Funkciju f zovemo* ***podintegralnom funkcijom****, $f(x)dx$* ***podintegralnim izrazom****, a interval $[a, b]$* ***područjem integracije****.*

Primjer 1 *a) Odredite približnu vrijednost $\int_0^1 x^2 dx$ koristeći ekvidistantnu subdiviziju za $n = 4$. Za međutočke koristite polovišta intervala subdivizije.*

b) Odredite donju i gornju ogradu (ocjenu) za $\int_0^1 x^2 dx$ koristeći subdiviziju pod a)

Rješenje:



a) *Ekvidistantna subdivizija znači da će svi podintervali biti jednako dugački i to upravo duljine* $h = \frac{1-0}{4} = \frac{1}{4}$ *pa vrijedi* $x_i = x_0 + i \cdot h$. *Stoga će subdivizija izgledati:*

$$D \ldots x_0 = 0 < x_1 = \frac{1}{4} < x_2 = \frac{1}{2} < x_3 = \frac{3}{4} < x_4 = 1$$

Međutočke su sredine podintervala pa imamo:

$$\overline{x}_1 = \frac{1}{8}, \quad \overline{x}_2 = \frac{3}{8}, \quad \overline{x}_3 = \frac{5}{8}, \quad \overline{x}_4 = \frac{7}{8}$$

Sada imamo:

$$S(D) = \frac{1}{4}\left(\frac{1}{8^2} + \frac{9}{8^2} + \frac{25}{8^2} + \frac{49}{8^2}\right) = \frac{1}{4} \cdot \frac{1}{8^2} \cdot 84 = \frac{21}{64}$$

b) *Ocjena odozdo (donja ograda):*

Za ocjenu odozdo moramo na svakom podintervalu pronaći minimum funkcije f *(koji sigurno postoji jer je* f *neprekidna funkcija, a podintervali su segmenti). S obzirom da je zadana podintegralna funkcija* $f(x) = x^2$ *na cijelom području integracije* $[0, 1]$ *rastuća funkcija, slijedi da se minimum postiže uvijek u lijevom rubu intervala tj. na intervalu* $[x_{i-1}, x_i]$ *minimum se postiže u* x_{i-1} *i iznosi* $f(x_{i-1}) = x_{i-1}^2$. *Slijedi:*

$$S(D) = \frac{1}{4}\left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right) = \frac{14}{64}$$

Ocjena odozgo (gornja ograda):

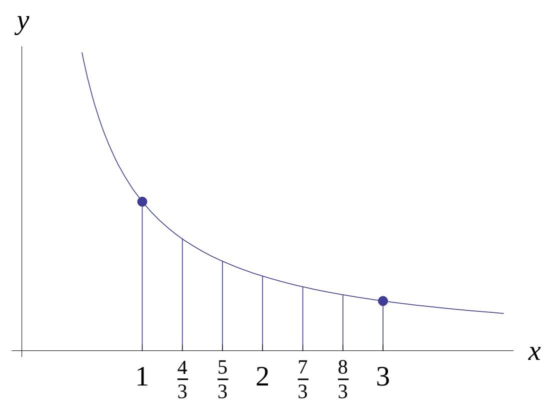
Analogno kao za ocjenu odozdo, ovdje moramo pronaći maksimum funkcije f na svim podintervalima. Maksimum opet sigurno u ovom slučaju postoji i na intervalu $[x_{i-1}, x_i]$ postiže se u x_i i iznosi $f(x_i) = x_i^2$. Slijedi:

$$S(D) = \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right) = \frac{30}{64}$$

Primjer 2 *a) Odredite približnu vrijednost $\int_1^3 \frac{1}{x}dx$ koristeći ekvidistantnu subdiviziju za $n = 6$. Za međutočke koristite polovišta intervala subdivizije.*

b) Odredite donju i gornju ogradu (ocjenu) $\int_1^3 \frac{1}{x}dx$ koristeći subdiviziju pod a)

RJEŠENJE:



a) Analogno kao u Primjeru 1. duljina svakog od podintervala će biti $h = \frac{3-1}{6} = \frac{1}{3}$ pa je subdivizija:

$$D \ldots x_0 = 1 < x_1 = \frac{4}{3} < x_2 = \frac{5}{3} < x_3 = 2 < x_4 = \frac{7}{3} < x_5 = \frac{8}{3} < x_6 = 3$$

Međutočke:

$$\overline{x}_1 = \frac{7}{6}, \quad \overline{x}_2 = \frac{9}{6}, \quad \overline{x}_3 = \frac{11}{6}, \quad \overline{x}_4 = \frac{13}{6}, \quad \overline{x}_5 = \frac{15}{6}, \quad \overline{x}_6 = \frac{17}{6}$$

Slijedi:

$$S(D) = \frac{1}{3}\left(\frac{6}{7} + \frac{6}{9} + \frac{6}{11} + \frac{6}{13} + \frac{6}{15} + \frac{6}{17}\right) = \frac{838192}{765765} = 1.09458$$

b) Analogno kao u Primjeru 1. potrebno je iskoristiti znanje da funkcija $f(x) = \frac{1}{x}$ pada na svojoj prirodnoj domeni. Detalje prepuštamo čitatelju.

Primjeri računanja određenih integrala po definiciji

Primjer 3

$$\int_a^b c\,dx = \lim_{m(D)\to 0} S(D) = \lim_{m(D)\to 0} \sum_{i=1}^{n} c(x_i - x_{i-1}) =$$

$$= \lim_{m(D)\to 0} c\left((x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})\right) = c(b-a)$$

Primjer 4 $\int_a^b xdx = ?$ *, međutočke su polovišta podintervala, tj.* $\overline{x}_i = \frac{x_{i-1}+x_i}{2}$

$$\int_a^b xdx = \lim_{m(D)\to 0} \sum_{i=1}^{n} \frac{x_{i-1}+x_i}{2}(x_i - x_{i-1}) = \lim_{m(D)\to 0} \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) = \frac{b^2}{2} - \frac{a^2}{2}$$

Primjer 5 $\int_a^b e^x dx \; = ?$ *, uzimamo ekvidistantnu subdiviziju s n točaka, dakle,* $h = \frac{b-a}{n}, x_i = a + ih$

$$\int_a^b e^x dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{a+ih}(a + ih - (a + (i-1)h)) = \lim_{n\to\infty} \sum_{i=1}^{n} h e^{a+ih} =$$

$$= e^a \lim_{n\to\infty} h \sum_{i=1}^{n} \left(e^h\right)^i = e^a \lim_{n\to\infty} \frac{b-a}{n} e^{\frac{b-a}{n}} \frac{1 - e^{(b-a)}}{1 - e^{\frac{b-a}{n}}} = (e^a - e^b) \lim_{n\to\infty} e^{\frac{b-a}{n}} \frac{\frac{b-a}{n}}{1 - e^{\frac{b-a}{n}}} =$$

$= \{$*čitatelju ostavljamo da izračuna sljedeće limese:*
$\lim_{n\to\infty} e^{\frac{b-a}{n}} = 1, \lim_{n\to\infty} \frac{\frac{b-a}{n}}{1-e^{\frac{b-a}{n}}} = -1\} = e^b - e^a$

Primjer 6 $\int_1^a \frac{dx}{x}$ *=?, gdje je* $a > 1$*. Uzimamo geometrijsku subdiviziju gdje je* $x_i = q^i = (a^{\frac{1}{n}})^i = a^{\frac{i}{n}} = \overline{x}_i$

$$\int_1^a \frac{dx}{x} = \lim_{m(D)\to 0} S(D) = \lim_{n\to\infty} \sum_{i=1}^{n} a^{-\frac{i}{n}}(a^{\frac{i}{n}} - a^{\frac{i-1}{n}}) = \lim_{n\to\infty} \sum_{i=1}^{n} (1 - a^{-\frac{1}{n}}) =$$

$$= \lim_{n\to\infty} (n - na^{-\frac{1}{n}}) = \lim_{n\to\infty} \frac{1 - a^{-\frac{1}{n}}}{\frac{1}{n}} = \{L'H\} = \lim_{n\to\infty} \frac{-a^{-\frac{1}{n}} \ln a \cdot \frac{1}{n^2}}{-\frac{1}{n^2}} = \ln a$$

Primjer 7 $\int_1^a x^\alpha dx$ *=?, gdje je* $a > 1$*. Ponovno kao i u (d) uzimamo geometrijsku subdiviziju za* $q = a^{\frac{1}{n}}, x_i = q^i = \overline{x}_i$

$$\int_1^a x^\alpha dx = \lim_{n\to\infty} \sum_{i=1}^{n} a^{\alpha\frac{i}{n}}(a^{\frac{i}{n}} - a^{\frac{i-1}{n}}) = \lim_{n\to\infty} \sum_{i=1}^{n} a^{(\alpha+1)\frac{i}{n}}(1 - a^{-\frac{1}{n}}) =$$

$$= \lim_{n\to\infty} (1 - a^{-\frac{1}{n}}) \sum_{i=1}^{n} (a^{\frac{\alpha+1}{n}})^i = \lim_{n\to\infty} (1 - a^{-\frac{1}{n}}) a^{\frac{\alpha+1}{n}} \frac{1 - (a^{\frac{\alpha+1}{n}})^n}{1 - a^{\frac{\alpha+1}{n}}} =$$

$$= (1 - a^{\alpha+1}) \lim_{n\to\infty} \frac{1 - a^{-\frac{1}{n}}}{1 - a^{\frac{\alpha+1}{n}}} = \{L'H\} = \frac{1 - a^{\alpha+1}}{-(\alpha+1)} = \frac{a^{\alpha+1} - 1}{\alpha+1}$$

1.3 Osnovna svojstva određenog integrala

1. $\int_a^b (\lambda f(x) + \mu g(x))\, dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx$ (**aditivnost i homogenost integrala**)
2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \forall c \in (a, b)$ (**aditivnost po području integracije**)
3. $\int_a^a f(x)dx = 0\,;\ \int_a^b f(x)dx = -\int_b^a f(x)dx$
4. $f(x) \leq g(x), \forall x \in [a, b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$
5. $f(x) > 0, \forall x \in [a, b] \Rightarrow \int_a^b f(x)dx > 0$

Koristeći navedena svojstva sada možemo po definiciji izvesti određeni integral $\int_a^b \frac{dx}{x}$ gdje je $0 < a < 1 < b$:

$$\int_a^b \frac{dx}{x} = \int_a^1 \frac{dx}{x} + \int_1^b \frac{dx}{x} = -\int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x} = \ln b - \ln a$$

Isto tako za $0 < a < 1 < b$ možemo izvesti određeni integral

$$\int_a^b x^\alpha dx = \int_a^1 x^\alpha dx + \int_1^b x^\alpha dx = -\int_1^a x^\alpha dx + \int_1^b x^\alpha dx = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1}$$

Primjer 8 *Neka je* $f(x) = \begin{cases} 1 & : x \in [2,5] \\ 0 & : x \notin [2,5] \end{cases}$ *te* $F(x) = \int_0^x f(t)dt$. *Odredite* $F(-1), F(2), F(4), F(10), F(x), \Gamma(f), \Gamma(F)$.

Rješenje:

$$\begin{aligned} F(-1) &= \int_0^{-1} f(t)dt = \int_0^{-1} 0dt = 0 \\ F(2) &= \int_0^2 f(t)dt = \int_0^2 0dt = 0 \\ F(4) &= \int_0^4 f(t)dt = \int_0^2 f(t)dt + \int_2^4 f(t)dt = \int_0^2 0dt + \int_2^4 1dt = 0 + (4-2) = 2 \\ F(10) &= \int_0^{10} f(t)dt = \int_0^2 0dt + \int_2^5 1dt + \int_5^{10} 0dt = 0 + (5-2) + 0 = 3 \end{aligned}$$

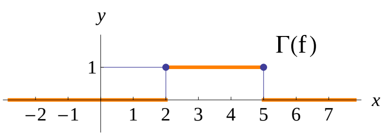
Sada ćemo odrediti $F(x)$ *za općeniti* x*:*

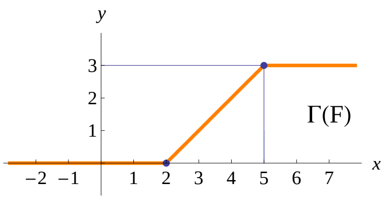
$x < 2$: $F(x) = \int_0^x f(t)dt = \int_0^x 0dt = 0$

$2 \le x \le 5$: $F(x) = \int_0^x f(t)dt = \int_0^2 f(t)dt + \int_2^x f(t)dt = \int_0^2 0dt + \int_2^x 1dt = x - 2$

$x > 5$: $F(x) = \int_0^x f(t)dt = \int_0^2 0dt + \int_2^5 1dt + \int_5^x 0dt = 0 + (5-2) + 0 = 3$

Pogledajmo kako izgleda graf funkcije f *i funkcije* F*:*





Primjer 9 *Neka je* $f(x) = \begin{cases} x & : x \in [0,3] \\ 0 & : x \notin [0,3] \end{cases}$ *te* $F(x) = \int_1^x f(t)dt$. *Odredite* $F(-1), F(1), F(2), F(12), F(x), \Gamma(f), \Gamma(F)$.

RJEŠENJE:

$$\begin{aligned}
F(-1) &= \int_1^{-1} f(t)dt = \int_1^0 tdt + \int_0^{-1} 0dt = -\frac{1}{2} + 0 = -\frac{1}{2} \\
F(1) &= \int_1^1 f(t)dt = 0 \\
F(2) &= \int_1^2 f(t)dt = \int_1^2 tdt = \frac{2^2 - 1^2}{2} = \frac{3}{2} \\
F(12) &= \int_1^{12} f(t)dt = \int_1^3 tdt + \int_3^{12} 0dt = \frac{3^2 - 1^2}{2} + 0 = 4
\end{aligned}$$

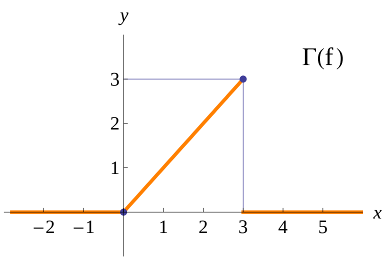
Sada ćemo odrediti $F(x)$ za općeniti x:

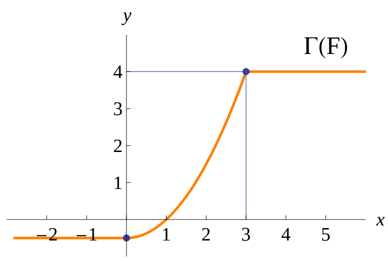
$x < 0$: $F(x) = \int_1^x f(t)dt = \int_1^0 tdt + \int_0^x 0dt = -\frac{1}{2}$

$0 \leq x \leq 3$: $F(x) = \int_1^x f(t)dt = \int_1^x tdt = \frac{x^2-1}{2}$

$x > 3$: $F(x) = \int_1^x f(t)dt = \int_1^3 tdt + \int_3^x 0dt = 4$

Pripadni grafovi funkcija f i F su:





1.4 Integralni teorem srednje vrijednosti

Neka je $f : [a, b] \to \mathbb{R}$ integrabilna funkcija i neka je $m \leq f(x) \leq M$ za svaki $x \in [a, b]$. Tada imamo:

$$m \leq f(x) \leq M \xRightarrow{\int_a^b} \int_a^b m\, dx \leq \int_a^b f(x)dx \leq \int_a^b M\, dx \iff$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \iff m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M$$

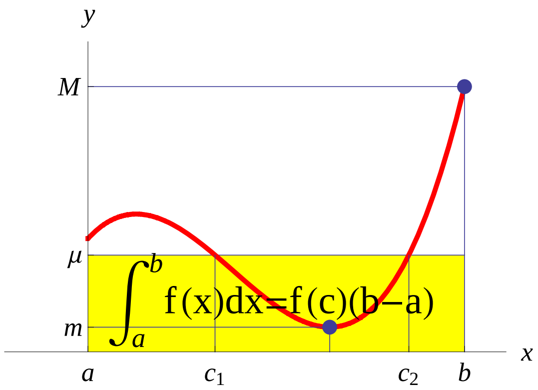
Broj

$$\mu = \frac{1}{b-a} \int_a^b f(x)dx$$

nazivamo **prosječna vrijednost (ili aritmetička sredina)** funkcije f na $[a, b]$.

Ako je f neprekidna na $[a, b]$, onda postoji $c \in [a, b]$ tako da je

$$f(c) = \mu = \frac{1}{b-a}\int_a^b f(x)dx \Longleftrightarrow f(c)(b-a) = \int_a^b f(x)dx$$



Primjer 10 *Koristeći teorem srednje vrijednosti ocjenite integrale:*

(a) $\int_0^4 \frac{dx}{x+2}$

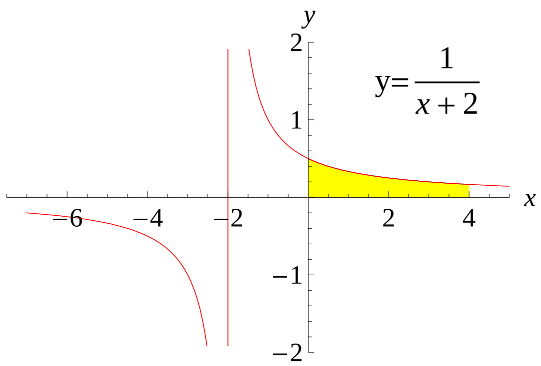
(b) $\int_0^3 \frac{2x+1}{x+1}dx$

(c) $\int_0^2 e^{x-x^2}dx$

RJEŠENJE:

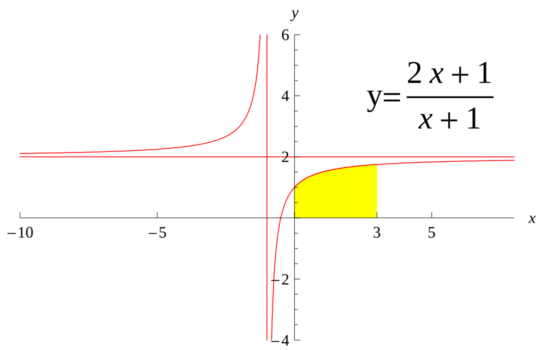
(a) *S obzirom da je podintegralna fukncija* $f(x) = \frac{1}{x+2}$ *padajuća, minimum se postiže u desnom rubu intervala tj.* $m = f(4) = \frac{1}{6}$, *a maksimum u lijevom rubu tj.* $M = f(0) = \frac{1}{2}$. *Slijedi:*

$$\frac{4}{6} \leq \int_0^4 \frac{dx}{x+2} \leq 2$$



(b) *Tražimo minimum i maksimum podintegralne funkcije* $f(x) = \frac{2x+1}{x+1}$. *S obzirom da je* $f'(x) = \frac{1}{(x+1)^2} > 0$ *slijedi da je* f *rastuća iz čega zaključujemo da se minimum postiže u lijevom rubu, a maksimum u desnom tj.* $m = f(0) = 1$, $M = f(3) = \frac{7}{4}$. *Slijedi:*

$$3 \leq \int_0^3 \frac{2x+1}{x+1} dx \leq \frac{21}{4}$$

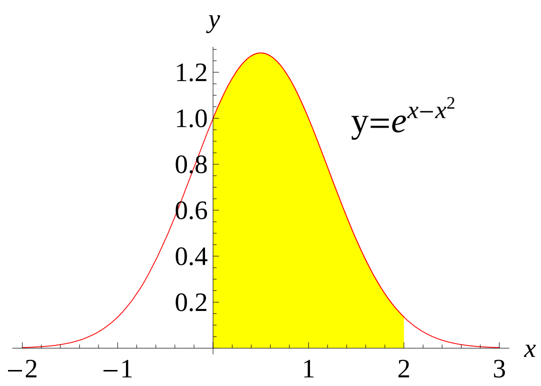


(c) *Tražimo minimum i maksimum podintegralne funkcije* $f(x) = e^{x-x^2}$. *S obzirom da je* $f'(x) = (1-2x)e^{x-x^2}$ *slijedi da je stacionarna točka* $x = \frac{1}{2}$. *Uspoređujemo vrijednost funkcije na rubovima i u stacionarnoj točki kako bi nasli minimum i maksimum:*

$$f(0) = 1, \quad f\left(\frac{1}{2}\right) = e^{\frac{1}{4}}, \quad f(2) = e^{-2} \Rightarrow m = e^{-2}, \quad M = e^{\frac{1}{4}}$$

Dakle, imamo:

$$2e^{-2} \leq \int_0^2 e^{x-x^2} dx \leq 2e^{\frac{1}{4}}$$



Primjer 11 *Odredite prosječnu vrijednost* μ *funkcije* f *na* $[a,b]$ *te odredite* $c \in [a,b]$ *tako da je* $f(c) = \mu$. *Nacrtajte pripadne grafove.*

(a) $f(x) = x^2$ *na* $[1,3]$

(b) $f(x) = \sqrt{x}$ *na* $[1,3]$

(c) $f(x) = \frac{1}{x}$ *na* $[1,3]$

(d) $f(x) = e^x$ *na* $[0,2]$

(e) $f(x) = \sqrt[3]{x}$ *na* $[-2,2]$

(f) $f(x) = x^2 - x$ *na* $[0,2]$

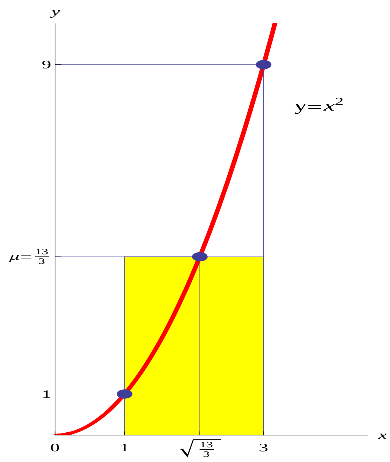
Rješenje:

(a) *Koristimo relaciju* $\int_a^b x^\alpha dx = \frac{b^{\alpha+1}-a^{\alpha+1}}{\alpha+1}$ *i definiciju prosječne vrijednosti* $\mu = \frac{1}{b-a}\int_a^b f(x)dx$

$$\mu = \frac{1}{3-1}\int_1^3 x^2 dx = \frac{13}{3}$$

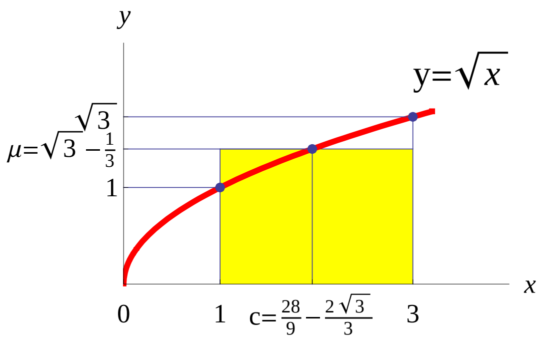
Sada tražimo $c \in [1,3]$ *takav da je* $f(c) = \mu = \frac{13}{3}$. *Slijedi*

$$c^2 = \frac{13}{3} \Rightarrow c = \sqrt{\frac{13}{3}}$$

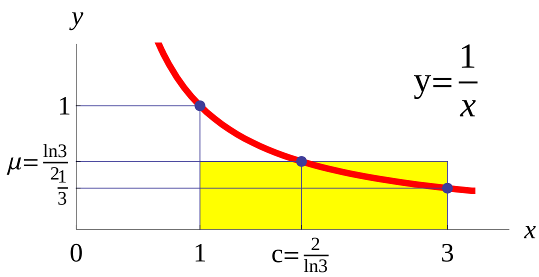


(b) $\mu = \frac{1}{3-1}\int_1^3 x^{\frac{1}{2}}dx = \sqrt{3} - \frac{1}{3} = 1.39872$

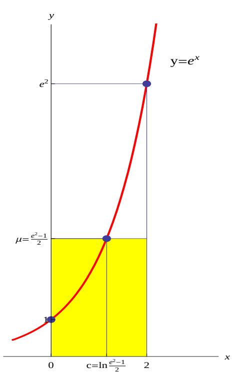
Tražimo $c \in [1,3]$ *takav da je* $f(c) = \mu \Rightarrow \sqrt{c} = 1.39872 \Rightarrow c = 1.95641$



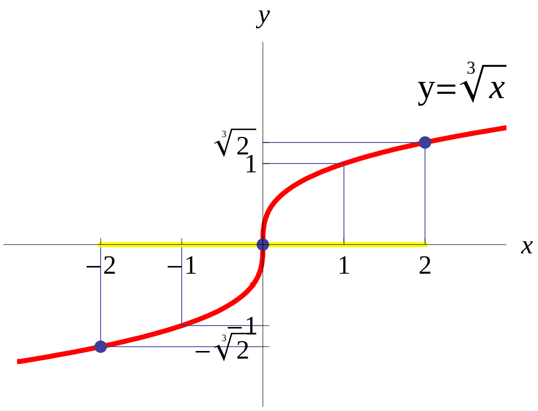
(c) $\mu = \frac{1}{2}\int_1^3 \frac{1}{x}dx = \frac{1}{2}\ln 3\ ,\ c = \frac{2}{\ln 3} = 1.82048$



(d) $\mu = \frac{1}{2}\int_0^2 e^x dx = \frac{e^2-1}{2}\ ,\ c = \ln\mu = 1.16144$



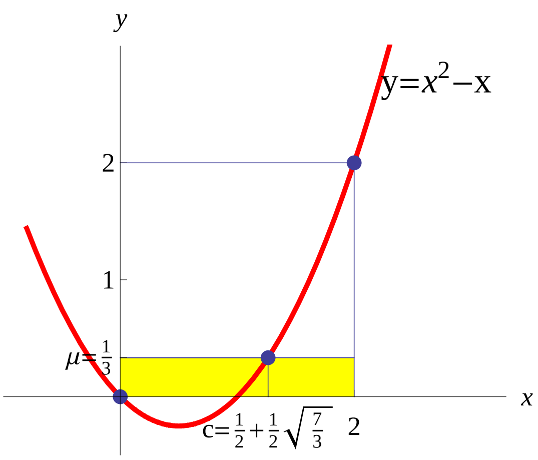
(e) $\mu = \frac{1}{4}\int_{-2}^{2} x^{\frac{1}{3}} dx = 0 \Rightarrow c = 0$



(f) $\mu = \frac{1}{2}\int_{0}^{2} (x^2 - x)dx = \frac{1}{2}\left(\int_{0}^{2} x^2 dx - \int_{0}^{2} x dx\right) = \frac{1}{3}$

Preostaje još odrediti $c \in [0,2]$ *takav da je* $f(c) = \mu$

$$c^2 - c = \frac{1}{3} \Rightarrow c_{1,2} = \frac{3 \pm \sqrt{21}}{6} \Rightarrow c = \frac{3 + \sqrt{21}}{6} = 1.26376$$



2 Pojam primitivne funkcije i neodređenog integrala. Neposredno integriranje.

2.1 Osnovni pojmovi, definicije i primjeri

Definicija 2 *Za funkciju $F : \langle a, b\rangle \to \mathbf{R}$ kažemo da je* **primitivna funkcija** *(antiderivacija) funkcije $f : \langle a, b\rangle \to \mathbf{R}$ ako je $F'(x) = f(x)$ za svaki $x \in \langle a, b\rangle$.*

Primjer 12 *(a)* $f(x) = x^2 \Longrightarrow F_1(x) = \frac{x^3}{3}\,, F_2(x) = \frac{x^3}{3} + \ln 2\,, F_3(x) = \frac{x^3}{3} + 10^6\,, \ldots$

(b) $f(x) = \sqrt{x} \Longrightarrow F_1(x) = \frac{2}{3}\sqrt{x^3}\,, F_2(x) = \frac{2}{3}\sqrt{x^3} + \sqrt{102}\,, \ldots$

(c) $f(x) = \frac{1}{x} \Longrightarrow F_1(x) = \ln|x|\,, \ldots$

(d) $f(x) = e^{3x} \Longrightarrow F_1(x) = \frac{e^{3x}}{3}\,, \ldots$

(e) $f(x) = \sin 2x \Longrightarrow F_1(x) = -\frac{1}{2}\cos 2x\,, \ldots$

Primjer 13 *(a) Neka je $f(x) = x^3$. Provjerite da su $F_1(x) = \frac{1}{4}x^4$, $F_2(x) = \frac{1}{4}x^4 + 1$, $F_3(x) = \frac{1}{4}x^4 + \sqrt[4]{\pi}$ primitivne funkcije funkcije f.*

(b) Neka je $f(x) = \frac{1}{x}$. Pokažite da je funkcija $F(x) = \ln(C|x|)$ primitivna funkcija funkcije f za svaki $C > 0$.

(c) Pokažite da je funkcija $F(x) = \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt{2}}\right)$ primitivna funkcija funkcije $f(x) = \frac{1}{1+\cos^2 x}$ za $x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbf{Z}$.

Problem jedinstvenosti primitivne funkcije

Teorem 1 *Neka su $F, G : \langle a, b\rangle \to \mathbb{R}$ primitivne funkcije funkcije $f : [a, b] \to \mathbb{R}$. Tada postoji $C \in \mathbb{R}$ tako da je $G(x) = F(x) + C$.*

Dokaz: Definiramo funkciju $H(x) = G(x) - F(x) \overset{\frac{d}{dx}}{\Longrightarrow} H'(x) = G'(x) - F'(x) = 0\,, \forall x \in \langle a, b\rangle$. Iz Lagrangeovog teorema srednje vrijednosti slijedi da je $H(x) = C$ za neki $C \in \mathbb{R}$ i za svaki $x \in \langle a, b\rangle$ tj. $G(x) = F(x) + C\,, \forall x \in \langle a, b\rangle$ $\square$

Definicija 3 *Skup svih primitivnih funkcija funkcije f zovemo* **neodređenim integralom** *i označavamo sa:*

$$\int f(x)dx = F(x) + C,$$

pri čemu je F bilo koja primitivna funkcija funkcije f. Kažemo da je $f(x)$ podintegralna funkcija, $f(x)dx$ podintegralni izraz, x varijabla integracije i C konstanta integracije.

Primjer 14 *(a)* $\int x^3 dx = \frac{x^4}{4} + C\,, C \in \mathbb{R}$

(b) $\int \frac{dx}{x+1} = \ln|x+1| + C\,, \in \mathbb{R}$

(c) $\int a^x dx = \frac{a^x}{\ln a} + C\,, C \in \mathbb{R}$

2.2 Osnovna svojstva neodređenog integrala. Neposredna integracija

Osnovna svojstva neodređenog integrala su:

1. $d\int f(x)dx = f(x)dx \Leftrightarrow \left(\int f(x)dx\right)' = f(x)$
   pr. $d\int \ln x dx = \ln x dx.$

2. $\int dF(x) = F(x) + C \Leftrightarrow \int F'(x)dx = F(x) + C$
   pr. $\int d(\sin x) = \sin x + C$

3. $\int kf(x)dx = k\int f(x)dx, \quad k \in \mathbf{R}$
   pr. $\int \frac{3}{x}dx = 3\int \frac{dx}{x} = 3\ln|x| + C$

4. $\int (f_1(x)+f_2(x))dx = \int f_1(x)dx + \int f_2(x)dx$
   pr. $\int (x^3+2^x-1)dx = \int x^3dx + \int 2^xdx - \int dx = \frac{x^4}{4} + \frac{2^x}{\ln 2} - x + C$

5. $$\int f(\phi(x))\phi'(x)\,dx = F(\phi(x)) + C \qquad (1)$$
   pri čemu je $F'(x) = f(x)$.

Metoda neposredne integracije sastoji se u tome da korištenjem gornjih osnovnih svojstava neodređenog integrala neke neodređene integrale svedemo na tablične.

Tablični integrali:

1. $\displaystyle\int dx = x + C$

2. $\displaystyle\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1$

3. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

4. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C, \quad a \neq 0$

5. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C, \quad a \neq 0$

6. $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) + C, \quad a \neq 0$

7. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C, \quad a \neq 0$

8. $\displaystyle\int a^x dx = \frac{a^x}{\ln a} + C$

9. $\displaystyle\int \sin x dx = -\cos x + C$

10. $\displaystyle\int \cos x dx = \sin x + C$

11. $\displaystyle\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$

12. $\displaystyle\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$

Zadatak 1 *Odredite neodređene integrale a)* $\int x^6\, dx$ *b)* $\int \frac{dx}{\sqrt{x}}$ *c)* $\int \sin(3x)dx$.

Rješenje: Deriviranjem desne strane jednakosti lako se provjeri da je a) $\int x^6\, dx = \frac{1}{7}x^7 + C$. b) $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$. c) $\int \sin(3x)dx = -\frac{1}{3}\cos(3x) + C$.

Zadatak 2 *Odredite neodređene integrale*

a) $\int e^{3x}\, dx$ *b)* $\int e^{5x+2}\, dx$ *c)* $\int \frac{dx}{x+2}$ *d)* $\int \frac{dx}{7x-3}$.

Rješenje:

$$\begin{aligned}
a)\quad & \int e^{3x}\, dx = \frac{1}{3}\int e^{3x}\, d(3x) = \frac{1}{3}e^{3x} + C \\
b)\quad & \int e^{5x+2}\, dx = \frac{1}{5}\int e^{5x+2}\, d(5x+2) = \frac{1}{5}e^{5x+2} + C \\
c)\quad & \int \frac{1}{x+2}\, dx = \int \frac{1}{x+2}\, d(x+2) = \ln|x+2| + C \\
d)\quad & \int \frac{1}{7x-3}\, dx = \frac{1}{7}\int \frac{1}{7x-3}\, d(7x-3) = \frac{1}{7}\ln|7x-3| + C
\end{aligned}$$

Zadatak 3

$$\int \frac{(1-x)^2}{x\sqrt{x}}dx = \int \frac{dx}{x\sqrt{x}} - 2\int \frac{x}{x\sqrt{x}}dx + \int \frac{x^2}{x\sqrt{x}} = -\frac{2}{\sqrt{x}} - 4\sqrt{x} + \frac{2}{3}x\sqrt{x} + C.$$

Zadatak 4

$$\int \frac{2x-3}{x^2-3x+5}dx = \left\{ \begin{array}{c} t = x^2 - 3x + 5 \\ dt = (2x-3)dx \end{array} \right\} = \int \frac{dt}{t} = \ln|t| + C = \ln|x^2 - 3x + 5| + C.$$

Je li potrebna apsolutna vrijednost?

Zadatak 5

$$\int \frac{\cos x}{(1+\sin x)^3}dx = \left\{ \begin{array}{c} t = 1 + \sin x \\ dt = \cos x dx \end{array} \right\} = \int \frac{dt}{t^3} = -\frac{1}{2t^2} + C = -\frac{1}{2(1+\sin x)^2} + C.$$

Zadatak 6 *(DZ)*

$$\begin{aligned}\int \frac{\cos 2x}{\sin^2 x \cos^2 x}dx &= \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}dx - \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}dx \\ &= \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = -\operatorname{ctg} x - \operatorname{tg} x + C.\end{aligned}$$

Zadatak 7

$$\int \frac{x^2\, dx}{1+x^2} = \int \frac{x^2+1-1}{1+x^2}dx = \int dx - \int \frac{dx}{1+x^2} = x - \operatorname{arctg} x + C.$$

Zadatak 8

$$\int \frac{dx}{5+x^2} = \frac{1}{5}\int \frac{dx}{1+\left(\frac{x}{\sqrt{5}}\right)^2} = \frac{\sqrt{5}}{5}\int \frac{d\left(\frac{x}{\sqrt{5}}\right)}{1+\left(\frac{x}{\sqrt{5}}\right)^2} = \frac{1}{\sqrt{5}}\operatorname{arctg}\frac{x}{\sqrt{5}} + C.$$

Zadatak 9 *(DZ)*

$$\int \frac{x^2}{1+x^6}dx = \left\{\begin{array}{c} t = x^3 \\ dt = 3x^2dx \end{array}\right\} = \int \frac{dt}{3(1+t^2)} = \frac{1}{3}\operatorname{arctg} t + C = \frac{1}{3}\operatorname{arctg}(x^3) + C.$$

Zadatak 10

$$\int 5^{2-3x}dx = \int 5^{2-3x}\left(-\frac{1}{3}d(2-3x)\right) = \left\{\begin{array}{c} t = 2-3x \\ dt = -3dx \end{array}\right\}$$

$$= -\frac{1}{3}\int 5^t dt = -\frac{1}{3}\frac{5^t}{\ln 5} + C = -\frac{1}{3}\frac{5^{2-3x}}{\ln 5} + C.$$

Zadatak 11

$$\int x\sqrt{2+x^2}dx = \left\{\begin{array}{c} t = 2+x^2 \\ dt = 2xdx \end{array}\right\} = \int \sqrt{t}\frac{dt}{2} = \frac{1}{2}\cdot\frac{2}{3}t\sqrt{t} + C = \frac{1}{3}(2+x^2)\sqrt{2+x^2} + C.$$

Zadatak 12

$$\int \frac{\arcsin x + x}{\sqrt{1-x^2}}dx = \int \frac{\arcsin x dx}{\sqrt{1-x^2}} + \int \frac{xdx}{\sqrt{1-x^2}}$$

$$= \int \arcsin x d \arcsin x - \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} d(1-x^2) = \frac{1}{2}\arcsin^2 x - \sqrt{1-x^2} + C.$$

Zadatak 13

$$\begin{aligned}\int \frac{dx}{\sin x} &= \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{dx}{2\operatorname{tg}\frac{x}{2}\cos^2\frac{x}{2}} = \int \frac{d(\operatorname{tg}\frac{x}{2})}{\operatorname{tg}\frac{x}{2}} \\ &= \int d\left(\ln\left|\operatorname{tg}\frac{x}{2}\right|\right) = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C.\end{aligned}$$

Zadatak 14 *(DZ) Analognim transformacijam kao u prethodnom zadatku odredite a)* $\int \frac{dx}{\cos x}$ *b)* $\int \frac{dx}{1+\sin x}$ *c)* $\int \frac{dx}{1+\cos x}$.

2.3 Metoda supstitucije

Korištenjem formule za derivaciju kompozicije funkcija i formule za derivaciju inverzne funkcije dobije se:

$$\int f(x)\,dx = \left\{ \begin{array}{c} x = \varphi(t) \\ dx = \varphi'(t)dt \end{array} \right\} = \int f(\varphi(t))\varphi'(t)\,dt.$$

Ili preciznije: Ako je $F(t)$ primitivna funkcija funkcije $f(\varphi(t))\varphi'(t)$, onda je $F(\varphi^{-1}(x))$ primitivna funkcija funkcije $f(x)$, odnosno $\int f(x)dx = F(\varphi^{-1}(x)) + C$ pri čemu je $F'(t) = f(\varphi(t))\varphi'(t)$.

Napomena: Usporedi formulu u metodi supstitucije sa (5) u osnovnim svojstvima neodređenog integrala.

Zadatak 15 *Riješite* $\int \frac{dx}{x\sqrt{1+x^2}}$ *supstitucijom a)* $x = \frac{1}{t}$ *b)* $x = \operatorname{tg} t$.

Rješenje:

a)
$$\int \frac{dx}{x\sqrt{1+x^2}} = \left\{ \begin{array}{c} x = \frac{1}{t} \\ dx = -\frac{1}{t^2}dt \end{array} \right\} = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{1+\frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{1+t^2}}$$

$$= -\ln\left(t + \sqrt{1+t^2}\right) + C = -\ln\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right) + C.$$

b)
$$\int \frac{dx}{x\sqrt{1+x^2}} = \left\{ \begin{array}{c} x = \operatorname{tg} t \Rightarrow t = \operatorname{arctg} x \\ dx = \frac{dt}{\cos^2 t} \end{array} \right\} = \int \frac{\frac{dt}{\cos^2 t}}{\operatorname{tg} t\sqrt{1+\operatorname{tg}^2 t}}$$

$$= \int \frac{dt}{\sin t} = \int \frac{d\left(\operatorname{tg}\frac{t}{2}\right)}{\operatorname{tg}\frac{t}{2}} = \ln\left|\operatorname{tg}\frac{t}{2}\right| + C = \ln\left|\operatorname{tg}\frac{\operatorname{arctg} x}{2}\right| + C.$$

□

Usporedite oblike primitivnih funkcija u prethodnom zadatku pod a) i b). S obzirom na Teorem 1 na str. 19. što zaključujete?

Zadatak 16 *Riješite korištenjem trig. supstitucije oblika* $x = a\sin t$ *a)* $\int \sqrt{2-x^2}dx$ *b) (DZ)* $\int \sqrt{5-x^2}dx$.

Rješenje:

a) $$\int \sqrt{2-x^2}dx = \left\{ \begin{array}{c} x=\sqrt{2}\sin t \Rightarrow t=\arcsin\frac{x}{\sqrt{2}} \\ dx=\sqrt{2}\cos t dt \end{array} \right\} = \int \sqrt{2-2\sin^2 t}\sqrt{2}\cos t\, dt$$

$$= 2\int \cos^2 t\, dt = \int (1+\cos 2t)dt = t + \frac{1}{2}\sin 2t + C$$

$$= t + \sin t\sqrt{1-\sin^2 t} + C = \arcsin\frac{x}{\sqrt{2}} + \frac{x}{\sqrt{2}}\sqrt{1-\frac{x^2}{2}} + C = \arcsin\frac{x}{\sqrt{2}} + \frac{1}{2}x\sqrt{2-x^2} + C.$$

b) $$\int \sqrt{5-x^2}dx = \left\{ \begin{array}{c} x=\sqrt{5}\sin t \Rightarrow t=\arcsin\frac{x}{\sqrt{5}} \\ dx=\sqrt{5}\cos t dt \end{array} \right\} = \int \sqrt{5-5\sin^2 t}\sqrt{5}\cos t\, dt$$

$$= 5\int \cos^2 t\, dt = 5\int \frac{1+\cos 2t}{2}dt = \frac{5}{2}\left(t + \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{5}{2}(t+\sin t\sqrt{1-\sin^2 t}) + C = \frac{5}{2}\left(\arcsin\frac{x}{\sqrt{5}} + \frac{x}{\sqrt{5}}\sqrt{1-\frac{x^2}{5}}\right) + C.$$

$\square$

2.4 Metoda parcijalne integracije

Integrirajući formulu za deriviranje produkta funkcija dobije se formula parcijalne integracije izražena u sljedećem teoremu.

Teorem 2 *Neka su f i g neprekidno derivabilne na $\langle a,b\rangle$. Tada vrijedi sljedeća jednakost neodređenih integrala*

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

Pokrata: U primjeni formula parcijalne integracije se najčešće zapisuje u diferencijalnom obliku:

$$\int udv = uv - \int vdu.$$

$$\texttt{OBLICI:} \qquad \int P_n(x) \left\{ \begin{array}{c} e^{ax} \\ \sin \alpha x \\ \cos \beta x \end{array} \right\} dx.$$

Zadatak 17 *Odredite a)* $\int x \sin(\pi x) dx$ *b)* $\int x^2 e^{-3x} dx$.

Rješenje:

$$a) \quad \int x \sin(\pi x) dx = \left\{ \begin{array}{c} u = x \Rightarrow du = dx \\ dv = \sin(\pi x) dx \Rightarrow v = -\frac{1}{\pi} \cos(\pi x) \end{array} \right\}$$

$$= -\frac{1}{\pi} x \cos(\pi x) + \frac{1}{\pi} \int \cos(\pi x) dx = -\frac{1}{\pi} x \cos(\pi x) + \frac{1}{\pi^2} \sin(\pi x) + C.$$

$$b) \quad \int x^2 e^{-3x} dx = \left\{ \begin{array}{c} u = x^2 \Rightarrow du = 2x dx \\ dv = e^{-3x} dx \Rightarrow v = -\frac{1}{3} e^{-3x} \end{array} \right\} = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx$$

$$= \left\{ \begin{array}{c} u = x \Rightarrow du = dx \\ dv = e^{-3x} dx \Rightarrow v = -\frac{1}{3} e^{-3x} \end{array} \right\} = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \left[-\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} dx \right]$$

$$= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{2}{27} e^{-3x} + C = -\frac{1}{3} e^{-3x} \left[x^2 + \frac{2}{3} x + \frac{2}{9} \right] + C.$$

□

Zadatak 18 *(DZ) Odredite* $\int \frac{x^2 - 3x + 5}{e^{4x}} dx$ $\quad \left(\ldots = -\frac{1}{32}(8x^2 - 20x + 35) e^{-4x} + C \right)$.

$$\texttt{OBLICI:} \qquad \int P_n(x) \left\{ \begin{array}{c} \ln(ax) \\ \arcsin(\alpha x) \\ \operatorname{arctg}(\beta x) \end{array} \right\} dx.$$

Zadatak 19 *Odredite a)* $\int x \ln 2x dx$ *b)* $\int \arcsin x dx$.

Rješenje:

$$a)\int x\ln(2x)dx=\left\{\begin{array}{c} u=\ln(2x)\Rightarrow du=\frac{dx}{x}\\ dv=xdx\Rightarrow v=\frac{x^2}{2}\end{array}\right\}=\frac{x^2}{2}\ln(2x)-\frac{1}{2}\int xdx=\frac{1}{2}x^2\ln(2x)-\frac{x^2}{4}+C.$$

$$b)\quad\int\arcsin x dx=\left\{\begin{array}{c} u=\arcsin x\Rightarrow du=\frac{dx}{\sqrt{1-x^2}}\\ dv=dx\Rightarrow v=x\end{array}\right\}=x\arcsin x-\int\frac{xdx}{\sqrt{1-x^2}}$$

$$=x\arcsin x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}d(1-x^2)=x\arcsin x+\sqrt{1-x^2}+C.$$

□

''CIKLIČKA'' PARCIJALNA INTEGRACIJA

Zadatak 20 *Koristeći cikličku integraciju odredite*

a) $\int\sqrt{x^2-1}\,dx$ *b)* $\int\sqrt{x^2+1}\,dx$ *c)* $\int\sqrt{1-x^2}\,dx$.

Rješenje:

$$\begin{aligned}
a)\quad I &= \int \sqrt{x^2-1}\,dx = \int \frac{x^2-1}{\sqrt{x^2-1}}\,dx = \int \frac{x^2}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx \\
&= \left\{ \begin{array}{ll} u = x & dv = \frac{xdx}{\sqrt{x^2-1}} \\ du = dx & v = \frac{1}{2}\int \frac{d(x^2-1)}{\sqrt{x^2-1}} = \frac{1}{2}\frac{(x^2-1)^{1/2}}{\frac{1}{2}} = \sqrt{x^2-1} \end{array} \right\} \\
&= x\sqrt{x^2-1} - \int \sqrt{x^2-1}\,dx - \ln|x+\sqrt{x^2-1}| \\
\Rightarrow \quad 2I &= x\sqrt{x^2-1} - \ln|x+\sqrt{x^2-1}| + C \\
I &= \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}| + C
\end{aligned}$$

$$\begin{aligned}
b)\quad I &= \int \sqrt{x^2+1}\,dx = \int \frac{x^2+1}{\sqrt{x^2+1}}\,dx = \int \frac{x^2}{\sqrt{x^2+1}}\,dx + \int \frac{1}{\sqrt{x^2+1}}\,dx \\
&= \left\{ \begin{array}{ll} u = x & dv = \frac{xdx}{\sqrt{x^2+1}} \\ du = dx & v = \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{x^2+1}} = \sqrt{x^2+1} \end{array} \right\} \\
&= x\sqrt{x^2+1} - \int \sqrt{x^2+1}\,dx + \ln(x+\sqrt{x^2+1}) \\
\Rightarrow \quad 2I &= x\sqrt{x^2+1} + \ln(x+\sqrt{x^2+1}) + C \\
I &= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln(x+\sqrt{x^2+1}) + C
\end{aligned}$$

$$\begin{aligned}
c)\quad I &= \int \sqrt{1-x^2}\,dx = \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x^2}{\sqrt{1-x^2}}\,dx + \\
&= \left\{ \begin{array}{ll} u = x & dv = \frac{xdx}{\sqrt{1-x^2}} \\ du = dx & v = -\frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\sqrt{1-x^2} \end{array} \right\} \\
&= \arcsin x - \left(-x\sqrt{1-x^2} + \int \sqrt{1-x^2}\,dx\right) \\
\Rightarrow \quad 2I &= \arcsin x + x\sqrt{1-x^2} + C \\
I &= \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C
\end{aligned}$$

NAPOMENA: zadatak pod c) najbolje je riješiti metodom iz Zadatka 16 □

Zadatak 21 *(DZ) Odredite a)* $\int e^x \sin x dx$ *b)* $\int \frac{x^2 dx}{\sqrt{1+x^2}}$.

Rješenje:

$$a) \quad I = \int e^x \sin x dx = \left\{ \begin{array}{c} u = \sin x \Rightarrow du = \cos x dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right\} = e^x \sin x - \int e^x \cos x dx$$

$$= \left\{ \begin{array}{c} u = \cos x \Rightarrow du = -\sin x dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right\} = e^x \sin x - \left[e^x \cos x + \int e^x \sin x dx \right]$$

$$= e^x \left[\sin x - \cos x \right] - I.$$

Dobije se $I = e^x(\sin x - \cos x) - I$, što daje $I = \frac{1}{2}e^x(\sin x - \cos x) + C$.

$$b) \quad I = \int \frac{x^2 dx}{\sqrt{1+x^2}} = \int x \frac{x dx}{\sqrt{1+x^2}} = \left\{ \begin{array}{c} u = x \Rightarrow du = dx \\ dv = \frac{x dx}{\sqrt{1+x^2}} \Rightarrow v = \sqrt{1+x^2} \end{array} \right\}$$

$$= x\sqrt{1+x^2} - \int \sqrt{1+x^2} dx = x\sqrt{1+x^2} - \int \frac{1+x^2}{\sqrt{1+x^2}} dx,$$

dobije se $I = x\sqrt{1+x^2} - \ln(x + \sqrt{1+x^2}) - I$, što daje $I = \frac{1}{2}(x\sqrt{1+x^2} - \ln(x + \sqrt{1+x^2})) + C$. □

2.5 Integriranje nekih klasa funkcija

2.5.1 Integriranje racionalnih funkcija

U ovom poglavlju ćemo razmotriti integriranje funkcija oblika $f(x) = \dfrac{P_n(x)}{Q_m(x)}$ gdje su $P_n(x)$ i $Q_m(x)$ polinomi stupnja n i m respektivno. Ako je $n < m$, onda f zovemo pravom racionalnom funkcijom.

Postupak integriranja:

1. Svođenje racionalnih funkcija na zbroj polinoma i prave racionalne funkcije (dijeljenje)

2. Rastav prave racionalne funkcije na zbroj parcijalnih razlomaka oblika $\dfrac{A}{(x-x_0)^k}$ i $\dfrac{Bx+C}{(x^2+px+q)^l}$ gdje je $p^2-4q<0$. (vidi postupak u zadacima)

3. Integriranje parcijalnih razlomaka

Zadatak 22 *Odredite: a)* $\int \frac{dx}{x+2}$ *b)* $\int \frac{dx}{(x+2)^5}$

Rješenje:

a) $\displaystyle\int \frac{dx}{x+2} = \ln|x+2| + C \ , \ C \in \mathbb{R}$

b) $\displaystyle\int \frac{dx}{(x+2)^5} = -\frac{1}{4}\cdot\frac{1}{(x+2)^4} + C \ , \ C \in \mathbb{R}$ □

Zadatak 23 *Odredite: a)* $\int \frac{dx}{x^2+6x+13}$ *b)* $\int \frac{dx}{(x^2+6x+13)^2}$

Rješenje:

a) $\displaystyle\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x+3)^2+4} = \frac{1}{2}\,\mathrm{arctg}\,\frac{x+3}{2} + C, \ , \ C \in \mathbb{R}$

b) $\displaystyle\int \frac{dx}{(x^2+6x+13)^2} = \int \frac{dx}{((x+3)^2+4)^2} = \int \frac{dt}{(t^2+4)^2} = \frac{1}{4}\int \frac{t^2+4-t^2}{(t^2+4)^2}dt = \frac{1}{4}\cdot\frac{1}{2}\,\mathrm{arctg}\,\frac{t}{2} - \frac{1}{4}I$

$$I = \int \frac{t^2}{(t^2+4)^2}dt = \int t\frac{t}{(t^2+4)^2}dt = \left\{ \begin{array}{c} u = t \Rightarrow du = dt \\ dv = \frac{tdt}{(t^2+4)^2} \Rightarrow v = -\frac{1}{2}\cdot\frac{1}{t^2+4} \end{array} \right\} = -\frac{1}{2}\cdot$$

$\frac{t}{t^2+4} + \frac{1}{2} \cdot \frac{1}{2} \operatorname{arctg} \frac{t}{2} + C$

Sada slijedi

$$\int \frac{dx}{(x^2+6x+13)^2} = \frac{1}{8} \operatorname{arctg} \frac{t}{2} - \frac{1}{4}\left[-\frac{1}{2}\frac{t}{t^2+4} + \frac{1}{4} \operatorname{arctg} \frac{t}{2}\right] + C = \frac{1}{16} \operatorname{arctg} \frac{x+3}{2} + \frac{1}{8}\frac{x+3}{x^2+6x+13} + C$$

□

Zadatak 24 *Odredite: a)* $\int \frac{2x+1}{x-1}dx$, *b)* $\int \frac{x^2}{x-1}dx$, *c)* $\int \frac{x^3}{(x-1)^2}dx$

Rješenje:

a) $\int \frac{2x+1}{x-1}\,dx = \int \frac{2(x-1)+3}{x-1}\,dx = \int 2\,dx + 3\int \frac{dx}{x-1} = 2x + 3\ln|x-1| + C.$

b) $\int \frac{x^2}{x-1}\,dx = \int \frac{x^2-1+1}{x-1}\,dx = \int \left(x+1+\frac{1}{x-1}\right)dx = \frac{1}{2}x^2 + x + \ln|x-1| + C.$

c) Dijeljenje polinoma:

$$\begin{array}{l} \left(\quad x^3\right) \div \left(x^2-2x+1\right) = x+2+\dfrac{3x-2}{x^2-2x+1} \\ \underline{-x^3+2x^2-x} \\ \qquad 2x^2-x \\ \qquad \underline{-2x^2+4x-2} \\ \qquad\qquad 3x-2 \end{array}$$

Rastav na parcijalne razlomke: $\frac{3x-2}{(x-1)^2} = \frac{3}{x-1} + \frac{1}{(x-1)^2}.$

$$\int \frac{x^3}{x^2-2x+1}\,dx = \int \left(x+2+\frac{3}{x-1}+\frac{1}{(x-1)^2}\right)dx$$
$$= \frac{1}{2}x^2 + 2x + 3\ln|x-1| - \frac{1}{(x-1)^2} + C.$$

□

Zadatak 25 *Odredite:*

a) $\int \frac{dx}{(x+2)(x-1)(x-3)}$

b) $\int \frac{dx}{(x+1)(x-2)^2}$

c) $\int \frac{dx}{(x+1)(x-1)^2(x-3)^3}$

Rješenje:

$$a) \int \frac{dx}{(x+2)(x-1)(x-3)} = \int \left(\frac{\frac{1}{15}}{x+2} + \frac{-\frac{1}{6}}{x-1} + \frac{\frac{1}{10}}{x-3} \right) dx =$$
$$= \frac{1}{15} \ln|x+2| - \frac{1}{6} \ln|x-1| + \frac{1}{10} \ln|x-3| + C.$$

b) Rastav na parcijalne razlomke:

$$\frac{1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \qquad \Big/ \cdot (x+1)(x-2)^2$$

$$1 = A(x-2)^2 + B(x+1)(x-2) + C(x+1) \qquad (*)$$

I. način: uvrštavanje raznih vrijednosti za x u $(*)$

$$\begin{array}{llll} x=-1 & \Longrightarrow \; 1 = 9A & & \Longrightarrow \; A = \frac{1}{9} \\ x=2 & \Longrightarrow \; 1 = 3C & & \Longrightarrow \; C = \frac{1}{3} \\ x=0 & \Longrightarrow \; 1 = 4A - 2B + C & & \Longrightarrow \; B = -\frac{1}{9} \end{array}$$

II. način: izjednačavanje koeficijenata polinoma s lijeve i desne strane od $(*)$

$$1 = x^2(A+B) + x(-4A - B + C) + 4A - 2B + C.$$

Rješavanje sustava

$$\begin{array}{rcrcrcr} 0 & = & A & + & B & & \\ 0 & = & -4A & - & B & + & C \\ 1 & = & 4A & - & 2B & + & C \end{array}$$

daje $A = \frac{1}{9}$, $B = -\frac{1}{9}$ i $C = \frac{1}{3}$. Konačno,

$$\int \frac{dx}{(x+1)(x-2)^2} = \int \left(\frac{\frac{1}{9}}{x+1} + \frac{-\frac{1}{9}}{x-2} + \frac{\frac{1}{3}}{(x-2)^2} \right) dx$$
$$= \frac{1}{9} \ln|x+1| - \frac{1}{9} \ln|x-2| - \frac{1}{3(x-2)} + C.$$

□

Zadatak 26 *Odredite:*

a) $\int \frac{3x+2}{x^2+3x+10}dx$

b) $\int \frac{3x+2}{x^2+3x-10}dx$

c) $\int \frac{3x+2}{(x^2+3x+10)^2}dx$

d) $\int \frac{3x+2}{(x^2+3x-10)^2}dx$

Rješenje:

$$a) \int \frac{3x+2}{x^2+3x+10}\,dx = \int \frac{3x+2}{(x+\frac{3}{2})^2+\frac{31}{4}}\,dx = \left\{ \begin{array}{c} t = x+\frac{3}{2} \\ dt = dx \end{array} \right\} = \int \frac{3(t-\frac{3}{2})+2}{t^2+\frac{31}{4}}\,dt$$

$$= 3\int \frac{t}{t^2+\frac{31}{4}}\,dt - \frac{5}{2}\int \frac{dt}{t^2+\frac{31}{4}} = \frac{3}{2}\int \frac{d(t^2+\frac{31}{4})}{t^2+\frac{31}{4}} - \frac{5}{2}\frac{1}{\sqrt{31/4}}\operatorname{arctg}\frac{t}{\sqrt{31/4}}$$

$$= \frac{3}{2}\ln|t^2+\frac{31}{4}| - \frac{5}{\sqrt{31}}\operatorname{arctg}\frac{t}{\sqrt{31/4}} + C = \frac{3}{2}\ln(x^2+3x+10) - \frac{5}{\sqrt{31}}\operatorname{arctg}\frac{2x+3}{\sqrt{31}} + C.$$

$$b) \int \frac{3x+2}{x^2+3x-10}\,dx = \int \frac{3x+2}{(x+5)(x-2)}\,dx = \int \Big(\frac{13}{7(x+5)} + \frac{8}{7(x-2)}\Big)\,dx$$
$$= \frac{13}{7}\ln|x+5| + \frac{8}{7}\ln|x-2| + C.$$

$$c) \int \frac{3x+2}{(x^2+3x+10)^2}\,dx = \int \frac{3(x+\frac{3}{2})-\frac{5}{2}}{\big((x+\frac{3}{2})^2+\frac{31}{4}\big)^2}\,dx = \left\{ \begin{array}{c} t = x+\frac{3}{2} \\ dt = dx \end{array} \right\}$$
$$= 3\underbrace{\int \frac{t}{(t^2+\frac{31}{4})^2}\,dt}_{I_1} - \frac{5}{2}\underbrace{\int \frac{1}{(t^2+\frac{31}{4})^2}\,dt}_{I_2} = (*)$$

$$I_1 = \int \frac{t}{(t^2+\frac{31}{4})^2}\,dt = \frac{1}{2}\int \frac{d(t^2+\frac{31}{4})}{(t^2+\frac{31}{4})^2} = \frac{1}{2}\cdot\frac{-1}{t^2+\frac{31}{4}}$$

$$I_2 = \int \frac{1}{(t^2+\frac{31}{4})^2}\,dt = \frac{4}{31}\int \frac{(t^2+\frac{31}{4}-t^2)}{(t^2+\frac{31}{4})^2}\,dt = \frac{4}{31}\Bigg[\int \frac{dt}{t^2+\frac{31}{4}} - \underbrace{\int \frac{t^2\,dt}{(t^2+\frac{31}{4})^2}}_{I_3}\Bigg]$$

$$I_3 = \int t \cdot \frac{t}{(t^2+\frac{31}{4})^2}\,dt = \left\{ \begin{array}{ccc} u = t & \Rightarrow & du = dt \\ dv = \frac{t}{(t^2+\frac{31}{4})^2}dt & \Rightarrow & v = (\text{vidi} \;\; I_1) = -\frac{1}{2}\cdot\frac{1}{t^2+\frac{31}{4}} \end{array} \right\}$$

$$= -\frac{1}{2}\cdot\frac{t}{t^2+\frac{31}{4}} + \int \frac{1}{t^2+\frac{31}{4}}\,dt$$

Sada slijedi,

$$I_2 = \frac{4}{31}\left[\int \frac{dt}{t^2+\frac{31}{4}} - I_3\right] = \frac{4}{31}\left[\frac{1}{2}\cdot\frac{t}{t^2+\frac{31}{4}} + \frac{1}{2}\int\frac{dt}{t^2+\frac{31}{4}}\right]$$

$$= \frac{2}{31}\cdot\frac{t}{t^2+\frac{31}{4}} + \frac{2}{31}\frac{1}{\sqrt{31/4}}\operatorname{arctg}\frac{t}{\sqrt{31/4}}$$

$$(*) = 3I_1 - \frac{5}{2}I_2 = -\frac{3}{2(t^2+\frac{31}{4})} - \frac{5t}{31(t^2+\frac{31}{4})} - \frac{10}{31\sqrt{31}}\operatorname{arctg}\frac{t}{\sqrt{31/4}} + C$$

$$= \frac{-5x-54}{31(x^2+3x+10)} - \frac{10}{31\sqrt{31}}\operatorname{arctg}\frac{2x+3}{\sqrt{31}} + C.$$

d) Rastav na parcijalne razlomke:

$$\frac{3x+2}{(x-2)^2(x+5)^2} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$A = -\frac{5}{343}, \quad B = -\frac{13}{49}, \quad C = \frac{5}{343}, \quad D = \frac{8}{49}.$$

$$\int \frac{3x+2}{(x^2+3x-10)^2}\,dx = -\frac{5}{343}\ln|x+5| + \frac{13}{49(x+5)} + \frac{5}{343}\ln|x-2| - \frac{8}{49(x-2)} + C.$$

□

Zadatak 27 *(DZ) Odredite:*

a) $\int \frac{x+1}{x^2+x+1}dx$

b) $\int \frac{x+1}{(x^2+x+1)^2}dx$

Zadatak 28 *(DZ) Odredite rastave na parcijalne razlomke za: a)* $f(x) = \frac{1}{(x-1)(x^2+x+1)}$ *b)* $f(x) = \frac{1}{(x-1)^2(x^2+x+1)}$ *c)* $f(x) = \frac{1}{(x-1)(x^2+x+1)^2}$

Rješenje:

b) $$\frac{1}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1}$$

$$A = -\frac{1}{3}, \quad B = \frac{1}{3}, \quad C = \frac{1}{3}, \quad D = \frac{1}{3}.$$

c) $$\frac{1}{(x-1)(x^2+x+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

$$A = \frac{1}{9}, \quad B = -\frac{1}{9}, \quad C = -\frac{2}{9}, \quad D = -\frac{1}{3}, \quad E = -\frac{2}{3}.$$

□

2.5.2 Integriranje trigonometrijskih izraza

Zadatak 29 *Odredite: a)* $\int \frac{dx}{5+\sin x+2\cos x}$ *b)* $\int \frac{\sin x}{5+\cos x}dx$

Rješenje:

$$a) \int \frac{dx}{5+\sin x+2\cos x} = \left\{ \begin{array}{c} t = \operatorname{tg}\frac{x}{2} \quad \Rightarrow \quad x = 2\operatorname{arctg} t \\ dx = \frac{2dt}{1+t^2} \quad \sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \end{array} \right\}$$

$$= \int \frac{\frac{2\,dt}{1+t^2}}{5+\frac{2t}{1+t^2}+2\frac{1-t^2}{1+t^2}} = \frac{2}{3}\int \frac{dt}{t^2+\frac{2}{3}t+\frac{7}{3}} = \frac{2}{3}\int \frac{d(t+\frac{1}{3})}{(t+\frac{1}{3})^2+\frac{20}{9}}$$

$$= \frac{2}{3}\frac{1}{\sqrt{20/9}}\operatorname{arctg}\frac{t+\frac{1}{3}}{\sqrt{20/9}} + C = \frac{1}{\sqrt{5}}\operatorname{arctg}\frac{3\operatorname{tg}\frac{x}{2}+1}{2\sqrt{5}} + C.$$

$$b) \int \frac{\sin x}{5+\cos x}\,dx = \left\{ \begin{array}{c} t = \cos x \\ dt = -\sin x\,dx \end{array} \right\} = -\int \frac{1}{5+t}\,dt$$

$$= -\ln|5+t| + C = -\ln(5+\cos x) + C.$$

□

Zadatak 30 *Odredite: a)* $\int \frac{dx}{5\sin^2 x+\sin x\cos x+\cos^2 x}$ *b)* $\int \operatorname{tg}^n x dx$ *c)* $\int \operatorname{ctg}^n x dx$

Rješenje:

$$a) \int \frac{dx}{5\sin^2 x+\sin x\cos x+\cos^2 x} = \int \frac{1}{(5\operatorname{tg}^2 x+\operatorname{tg} x+1)}\frac{dx}{\cos^2 x}$$
$$= \left\{ \begin{array}{c} t=\operatorname{tg} x \\ dt=\frac{dx}{\cos^2 x} \end{array} \right\} = \int \frac{1}{5t^2+t+1}\,dt = \frac{1}{5}\int \frac{d(t+\frac{1}{10})}{(t+\frac{1}{10})^2+\frac{19}{100}}$$
$$= \frac{1}{5}\frac{1}{\sqrt{19/100}}\operatorname{arctg}\frac{t+\frac{1}{10}}{\sqrt{19/100}}+C = \frac{2}{\sqrt{19}}\operatorname{arctg}\frac{10\operatorname{tg} x+1}{\sqrt{19}}+C.$$

$$b) \text{ Za } n=3: \int \operatorname{tg}^3 x\,dx = \int \frac{1-\cos^2 x}{\cos^3 x}\sin x\,dx = \left\{ \begin{array}{c} t=\cos x \\ dt=-\sin x\,dx \end{array} \right\}$$
$$= -\int \frac{1-t^2}{t^3}\,dt = -\int t^{-3}\,dt + \int \frac{dt}{t} = \frac{1}{2}t^{-2}+\ln|t|+C = \frac{1}{2\cos^2 x}+\ln|\cos x|+C.$$

□

Zadatak 31 *Odredite: a)* $\int \cos^2 x dx$ *b)* $\int \sin^4 x dx$ *c)* $\int \cos^4 x dx$ *d)* $\int \cos^4 x\sin^3 x dx$

Rješenje:

$$b) \int \sin^4 x\,dx = \int \Big(\frac{1-\cos(2x)}{2}\Big)^2 dx = \frac{1}{4}\int \big(1-2\cos(2x)+\cos^2(2x)\big)\,dx$$
$$= \frac{1}{4}x-\frac{1}{4}\sin(2x)+\frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx = \frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C.$$

$$d) \int \cos^4 x\sin^3 x\,dx = \int \cos^4 x\big(1-\cos^2 x\big)\sin x\,dx = \left\{ \begin{array}{c} t=\cos x \\ dt=-\sin x\,dx \end{array} \right\}$$
$$= -\int t^4(1-t^2)\,dt = -\frac{1}{5}t^5+\frac{1}{7}t^7+C = -\frac{1}{5}\cos^5 x+\frac{1}{7}\cos^7 x+C.$$

□

Zadatak 32 *Odredite: a)* $\int \sin(2x)\cos(5x)dx$ *b)* $\int \cos(2x)\cos(5x)dx$ *c)* $\int \sin(2x)\sin(5x)dx$

Rješenje:

$$a) \int \sin(2x)\cos(5x)dx = \int \frac{1}{2}\big(\sin(2x+5x)+\sin(2x-5x)\big)\,dx$$
$$= \frac{1}{2}\int \big(\sin(7x)-\sin(3x)\big)\,dx = -\frac{1}{14}\cos(7x)+\frac{1}{6}\cos(3x)+C.$$

□

Zadatak 33 *(DZ) Odredite:* $\int \frac{dx}{4-3\cos^2 x+5\sin^2 x}$

Zadatak 34 *(DZ) Odredite:* $\int \sin^5 x\sqrt[3]{\cos x}dx$

2.5.3 Integriranje korijenskih izraza

Zadatak 35 *Odredite: a)* $\int \sqrt[3]{\frac{x+1}{x-1}}dx$ *b)* $\int \frac{dx}{\sqrt{x+1}+\sqrt[3]{x+1}}$

Rješenje:

$$b) \int \frac{dx}{\sqrt{x+1}+\sqrt[3]{x+1}} = \left\{\begin{array}{c} x+1=t^6 \\ dx=6t^5\,dt \end{array}\right\} = \int \frac{6t^5\,dt}{t^3+t^2} = 6\int \frac{t^3}{t+1}\,dt$$
$$= 6\int \left(t^2-t+1-\frac{1}{t+1}\right)dt = 2t^3-3t^2+6t-\ln|t+1|+C$$
$$= 2\sqrt{x+1}-3\sqrt[3]{x+1}+6\sqrt[6]{x+1}-\ln|\sqrt[6]{x+1}+1|+C.$$

□

Zadatak 36 *Odredite:* $\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}dx$

Rješenje:

$$\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}\,dx = \left\{\begin{array}{c} x=t^6 \\ dx=6t^5\,dt \end{array}\right\} = 6\int \frac{(t^3-1)t^5}{t^2+1}\,dt =$$
$$= 6\int \left(t^6-t^4-t^3+t^2-1+\frac{-t+1}{t^2+1}\right)dt = \cdots = \frac{6}{7}x\sqrt[6]{x}-\frac{6}{5}\sqrt[6]{x^5}$$
$$-\frac{3}{2}\sqrt[3]{x^2}+2\sqrt{x}+3\sqrt[3]{x}-6\sqrt[6]{x}+3\ln|\sqrt[3]{x}+1|+6\,\mathrm{arctg}\,(\sqrt[3]{x})+C.$$

□

Zadatak 37 *Odredite:* $\int x\sqrt{\frac{x-1}{x+1}}dx$

Rješenje:

$$\int x\sqrt{\frac{x-1}{x+1}}dx = \left\{ \begin{array}{c} t^2 = \frac{x-1}{x+1} = 1 - \frac{2}{x+1} \Rightarrow x = \frac{2}{1-t^2} - 1 = \frac{1+t^2}{1-t^2} \\ dx = \frac{4t}{(1-t^2)^2}dt \end{array} \right\}$$

$$= \int \frac{1+t^2}{1-t^2} \cdot t \cdot \frac{4t}{(1-t^2)^2} = 4\int \frac{t^4+t^2}{(1-t^2)^3}dt = 4\int \frac{t^4+t^2}{(1-t)^3(1+t)^3}dt = (*)$$

Rastav na parcijalne razlomke:

$$\frac{t^4+t^2}{(1-t)^3(1+t)^3} = \frac{\frac{1}{8}}{1-t} + \frac{-\frac{3}{8}}{(1-t)^2} + \frac{\frac{1}{4}}{(1-t)^3} + \frac{\frac{1}{8}}{1+t} + \frac{-\frac{3}{8}}{(1+t)^2} + \frac{\frac{1}{4}}{(1+t)^3}$$

$$(*) = 4\left[\frac{1}{8}\ln|1-t| - \frac{3}{8}\frac{1}{1-t} - \frac{1}{4}\frac{1}{2(1-t)^2} + \frac{1}{8}\ln|1+t| + \frac{3}{8}\frac{1}{1+t} - \frac{1}{4}\frac{1}{2(1+t)^2}\right]$$

$$= \frac{1}{2}\ln\left|1-\sqrt{\frac{x-1}{x+1}}\right| - \frac{3}{2}\frac{1}{1-\sqrt{\frac{x-1}{x+1}}} - \frac{1}{2\left(1-\sqrt{\frac{x-1}{x+1}}\right)^2} + \frac{1}{2}\ln\left|1+\sqrt{\frac{x-1}{x+1}}\right|$$

$$+\frac{3}{2}\frac{1}{1+\sqrt{\frac{x-1}{x+1}}} + \frac{1}{2\left(1+\sqrt{\frac{x-1}{x+1}}\right)^2}$$

□

Binomni integrali - integrali oblika $\int x^m(a+bx^n)^p\,dx$ gdje su m,n i p racionalni brojevi.

Zadatak 38 *Odredite:* $\int x^3(1+2x^2)^{-\frac{3}{2}}dx$

Rješenje:

$$\int x^3(1+2x^2)^{-\frac{3}{2}}dx = \left\{ \begin{array}{c} t = x^2 \Rightarrow x = \sqrt{t} \\ dx = \frac{1}{2\sqrt{t}}dt \end{array} \right\} = \frac{1}{2}\int t^{\frac{3}{2}}(1+2t)^{-\frac{3}{2}}t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\int t(1+2t)^{-\frac{3}{2}}dt = \left\{ \begin{array}{c} u^2 = 1+2t \Rightarrow t = \frac{u^2-1}{2} \\ dt = udu \end{array} \right\} = \frac{1}{2}\int \frac{u^2-1}{2}u^{-3}udu$$

$$= \frac{1}{4}\int (1-u^{-2})du = \frac{1}{4}u + \frac{1}{4u} + C = \frac{1}{4}\sqrt{1+2t} + \frac{1}{4\sqrt{1+2t}} + C$$

$$= \frac{1}{4}\sqrt{1+2x^2} + \frac{1}{4\sqrt{1+2x^2}} + C$$

□

Zadatak 39 *Odredite:* $\int \frac{dx}{x^2(2+x^3)^{\frac{5}{3}}}$

Rješenje:

$$\int \frac{dx}{x^2(2+x^3)^{\frac{5}{3}}} = \int x^{-2}(2+x^3)^{-\frac{5}{3}}dx = \left\{ \begin{array}{c} t = x^3 \Rightarrow dt = 3x^2dx \\ x = t^{\frac{1}{3}} \end{array} \right\} =$$

$$= \frac{1}{3}\int t^{-\frac{4}{3}}(2+t)^{-\frac{5}{3}}dt = \frac{1}{3}\int t^{-3}t^{\frac{5}{3}}(2+t)^{-\frac{5}{3}}dt = \frac{1}{3}\int t^{-3}\left(\frac{2+t}{t}\right)^{-\frac{5}{3}}dt =$$

$$= \left\{ \begin{array}{c} u^3 = \frac{2+t}{t} \Rightarrow 3u^2du = -\frac{2}{t^2}dt \\ t = \frac{2}{u^3-1} \end{array} \right\} = -\frac{1}{2}\int \frac{u^3-1}{2}\cdot u^{-3}\,du = -\frac{1}{4}\int 1-u^{-3}\,du =$$

$$= -\frac{1}{4}u - \frac{1}{8}u^{-2} = -\frac{1}{4}\sqrt[3]{\frac{2+t}{t}} - \frac{1}{8}\left(\frac{2+t}{t}\right)^{-\frac{2}{3}} = -\frac{1}{4}\sqrt[3]{\frac{2+x^3}{x^3}} - \frac{1}{8}\left(\frac{x^3}{2+x^3}\right)^{\frac{2}{3}}$$

□

Zadatak 40 *Promotrite: a)* $\int \frac{dx}{\sqrt{1+x^4}}$ *b)* $\int \frac{dx}{\sqrt[4]{1+x^4}}$ *c)* $\int \sqrt[4]{1+x^4}dx$

Rješenje: Integrali pod a) i c) su eliptički integrali, a pod b) je "krvava ruža". □

Zadatak 41 *Odredite: a)* $\int \frac{3x-1}{\sqrt{x^2-4x+5}}dx$ *b)* $\int \frac{2x-3}{\sqrt{1-x-x^2}}dx$

Rješenje:

a)

$$\begin{aligned}
\int \frac{3x-1}{\sqrt{x^2-4x+5}dx} &= \int \frac{d(x^2-4x+5)\cdot\frac{3}{2}+5dx}{\sqrt{x^2-4x+5}} \\
&= 3\sqrt{x^2-4x+5}+5\int\frac{dx}{\sqrt{(x-2)^2+1}} \\
&= 3\sqrt{x^2-4x+5}+5\ln\left(x-2+\sqrt{x^2-4x+5}\right)+C
\end{aligned}$$

b)

$$\begin{aligned}
\int \frac{2x-3}{\sqrt{1-x-x^2}dx} &= \int \frac{d(1-x-x^2)\cdot(-1)-4dx}{\sqrt{1-x-x^2}} \\
&= -2\sqrt{1-x-x^2}-4\int\frac{dx}{\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}} \\
&= -2\sqrt{1-x-x^2}-4\arcsin\frac{x+\frac{1}{2}}{\sqrt{\frac{5}{4}}}+C
\end{aligned}$$

□

Zadatak 42 *Odredite: a)* $\int \frac{3x^2-5x}{\sqrt{3-2x-x^2}}dx$ *b)* $\int \frac{-2x^2-2x}{\sqrt{3-2x-x^2}}dx$

Rješenje:

a)

$$\begin{aligned}
&\int \frac{3x^2-5x}{\sqrt{3-2x-x^2}}dx = \int\frac{3x^2-5x}{\sqrt{4-(x+1)^2}}dx = \{t=x+1\} = \int\frac{3t^2-11t+8}{\sqrt{4-t^2}}dt \\
&= 3\int t\frac{t}{\sqrt{4-t^2}} - 11\int\frac{d(4-t^2)\cdot(-\frac{1}{2})}{\sqrt{4-t^2}}+8\int\frac{dt}{\sqrt{4-t^2}} \\
&= \text{parcijalna integracija na prvom integralu} \\
&= -t\sqrt{4-t^2}+\int\sqrt{4-t^2}dt+11\sqrt{4-t^2}+8\arcsin\frac{t}{2} \\
&= -t\sqrt{4-t^2}+2\arcsin\frac{t}{2}+t\sqrt{1-\frac{t^2}{4}}+11\sqrt{4-t^2}+8\arcsin\frac{t}{2} \\
&= -(x+1)\sqrt{4-(x+1)^2}+2\arcsin\frac{x+1}{2}+t\sqrt{1-\frac{(x+1)^2}{4}} \\
&\quad +11\sqrt{4-(x+1)^2}+8\arcsin\frac{x+1}{2}
\end{aligned}$$

Čitatelju ostavljamo za riješiti $\int \sqrt{4-t^2}dt$.

$\square$

Zadatak 43 *Odredite: a)* $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4x-4}}$ *b)* $\displaystyle\int \frac{dx}{(x-1)^2\sqrt{x^2+x+1}}$

Rješenje:

a)

$$\begin{aligned}
&\int \frac{dx}{x^2\sqrt{x^2+4x-4}} = \left\{ \begin{array}{c} t=\frac{1}{x} \Rightarrow x=\frac{1}{t} \\ dt=-\frac{1}{x^2}dx \end{array} \right\} = -\int \frac{dt}{\sqrt{\frac{1}{t^2}+\frac{4}{t}-4}} \\
= & -\int \frac{dt}{\sqrt{\frac{1+4t-4t^2}{t^2}}} = -\int \frac{t}{\sqrt{1+4t-4t^2}}dt = -\int \frac{d(1+4t-4t^2)\cdot(-\frac{1}{8})+\frac{1}{2}dt}{\sqrt{1+4t-4t^2}} \\
= & \frac{1}{4}\sqrt{1+4t-4t^2} - \frac{1}{2}\int \frac{dt}{\sqrt{2-(2t-1)^2}} = \frac{1}{4}\sqrt{1+4t-4t^2} - \frac{1}{4}\arcsin\frac{2t-1}{\sqrt{2}} + C \\
= & \frac{1}{4}\sqrt{1+\frac{4}{x}-\frac{4}{x^2}} - \frac{1}{4}\arcsin\frac{\frac{2}{x}-1}{\sqrt{2}} + C
\end{aligned}$$

$\square$

Zadatak 44 *Odredite: a)* $\int \sqrt{x^2-2x-1}dx$ *b)* $\int \sqrt{1-4x-x^2}dx$ *c)* $\int \sqrt{x^2+2x+2}dx$

Rješenje:

a) Racionalizacijom svodimo na integral oblika kao u zadatku 42.

$$\begin{aligned}
&\int \sqrt{x^2-2x-1}dx = \int \frac{x^2-2x-1}{\sqrt{x^2-2x-1}}dx = \int \frac{(x-1)^2-2}{\sqrt{(x-1)^2-2}}dx \\
= & \left\{ \begin{array}{c} t=x-1 \Rightarrow x=t+1 \\ dt=dx \end{array} \right\} = \int \frac{t^2-2}{\sqrt{t^2-2}}dt = \int t\frac{t}{\sqrt{t^2-2}}dt - 2\int \frac{dt}{\sqrt{t^2-2}} \\
= & \left\{ \begin{array}{c} u=t \Rightarrow du=dt \\ v=\int \frac{d(t^2-2)\cdot\frac{1}{2}}{\sqrt{t^2-2}} = \sqrt{t^2-2} \end{array} \right\} = t\sqrt{t^2-2} - \int \sqrt{t^2-2}dt - 2\ln\left|t+\sqrt{t^2-2}\right|
\end{aligned}$$

Ako označimo s $I = \int \sqrt{t^2-2}dt = \int \sqrt{x^2-2x-1}dx$ onda imamo

$$\begin{aligned}
2I &= t\sqrt{t^2-2} - 2\ln\left|t+\sqrt{t^2-2}\right| \\
I &= \frac{1}{2}t\sqrt{t^2-2} - \ln\left|t+\sqrt{t^2-2}\right| + C \\
I &= \frac{1}{2}(x-1)\sqrt{x^2-2x-1} - \ln\left|x-1+\sqrt{x^2-2x-1}\right| + C
\end{aligned}$$

b)

$$\int \sqrt{1-4x-x^2}dx = \int \sqrt{5-(x+2)^2}dx = \{t = x+2\} = \int \sqrt{5-t^2}dt = \dots$$

c)

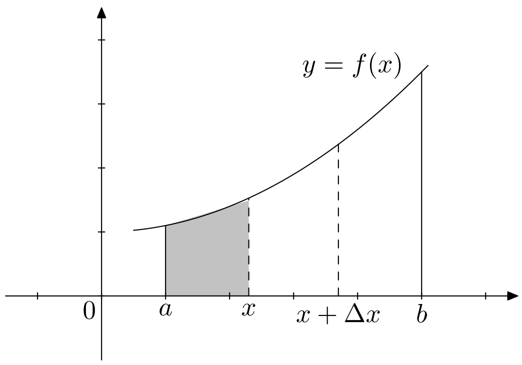
$$\begin{aligned}
&\int \sqrt{x^2+2x+2}dx = \int \sqrt{(x+1)^2+1}dx = \{t = x+1\} = \int \sqrt{t^2+1}dt \\
&= \int \frac{t^2+1}{\sqrt{t^2+1}}dt = \int t\frac{t}{\sqrt{t^2+1}}dt + \int \frac{dt}{\sqrt{t^2+1}} = \dots = \\
&= \frac{1}{2}(x+1)\sqrt{x^2+2x+2} + \frac{1}{2}\ln\left(x+1+\sqrt{x^2+2x+2}\right) + C
\end{aligned}$$

□

Zadatak 45 *Odredite: a)* $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ *b)* $\int \frac{dx}{(1-x^2)\sqrt{1+x^2}}$ *c)* $\int \frac{dx}{(1+x^2)\sqrt{x^2-1}}$

3 Fundamentalni teorem integralnog računa. Newton-Leibnizova formula

Integrali s promijenjivom granicom



$$x \mapsto P(x) = \int_a^x f(t)dt\,, \quad x \in [a, b]$$

Teorem 3 *Neka je $f : [a, b] \to \mathbb{R}$ neprekidna funkcija. Tada je $P(x) = \int_a^x f(t)dt$ primitivna funkcija funkcije f tj. $P'(x) = f(x)$. Ako je F bilo koja primitivna funkcija funkcije f, onda vrijedi*

$$\int_a^b f(x)dx = F(b) - F(a) \overset{ozn.}{=} F(x)\Big|_a^b$$

Dokaz:

$$P'(x) = \lim_{\Delta x \to 0} \frac{P(x + \Delta x) - P(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x} =$$

Sada koristimo integralni teorem srednje vrijednosti:
$\int_a^b f(x)dx = f(a + \gamma \cdot (b - a))(b - a)$ za $\gamma \in \langle 0, 1\rangle$.

$$= \lim_{\Delta x \to 0} \frac{f(x + \gamma \Delta x)\Delta x}{\Delta x} = f(x)\ ,\ \forall x$$

Iz toga da su $P(x)$ i $F(x)$ dvije primitivne funkcije i po teoremu o vezi između primitivnih funkcija (Teorem 1., str.19.) postoji $C \in \mathbb{R}$ t.d. je $P(x) = F(x) + C$ tj. $\int_a^x f(t)dt = F(x) + C$.
Uvrštavanjem da je $x = a$ slijedi $0 = \int_a^a f(t)dt = F(a) + C$ pa smo dobili da je konstanta $C = -F(a)$. Sada imamo $\int_a^x f(t)dt = F(x) - F(a)$ pa specijalno za $x = b$ dobijemo

$$\int_a^b f(t)dt = F(b) - F(a).$$

□

Iz prethodnog teorema znamo $\left(\int_a^x f(t)dt\right)' = f(x)$, tj. funkcija $F(x) = \int_a^x f(t)dt$ je primitivna funkcija funkcije f. Analogno,

$$F(\varphi(x)) = \int_a^{\varphi(x)} f(t)dt \overset{\frac{d}{dx}}{\Longrightarrow} \left(\int_a^{\varphi(x)} f(t)dt\right)' = (F(\varphi(x)))' = f(\varphi(x))\cdot\varphi'(x).$$

Isto tako, $G(x) = \int_x^a f(t)dt \overset{\frac{d}{dx}}{\Longrightarrow} G'(x) = -f(x)$ te

$$G(\psi(x)) = \int_{\psi(x)}^a f(t)dt \overset{\frac{d}{dx}}{\Longrightarrow} \left(\int_{\psi(x)}^a f(t)dt\right)' = -f(\psi(x)) \cdot \psi'(x).$$

Sada iz prethodna dva izvoda zaključujemo:

$$H(x) = \int_{\psi(x)}^{\varphi(x)} f(t)dt = \int_{\psi(x)}^a f(t)dt + \int_a^{\varphi(x)} f(t)dt \overset{\frac{d}{dx}}{\Longrightarrow}$$

$$\left(\int_{\psi(x)}^{\varphi(x)} f(t)dt\right)' = f(\varphi(x)) \cdot \varphi'(x) - f(\psi(x)) \cdot \psi'(x)$$

Primjer 15 *Pokažite da je* $F(x) = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{\operatorname{tg} x}{\sqrt{2}}$ *primitivna funkcija funkcije* $f(x) = \frac{1}{1+\cos^2 x}$ *te izračunajte* $\int_0^\pi \frac{dx}{1+\cos^2 x}$

Rješenje:
$$F'(x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{1 + \frac{\operatorname{tg}^2 x}{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\cos^2 x} = \frac{1}{2\cos^2 x + \sin^2 x} = \frac{1}{1 + \cos^2 x}$$

Sada kada znamo kako izgleda primitivna funkcija možemo lagano izračunati

zadani integral:
$\int_0^\pi \frac{dx}{1+\cos^2 x} = F(x)\Big|_0^\pi = 0$
Primijetimo da je $f(x) = \frac{1}{1+\cos^2 x} > 0$, $\forall x \in \mathbb{R}$, a ipak smo dobili kao rezultat u prethodnom integralu nula. Razlog tome je što se unutar područja integracije nalazi točka koja nije u domeni funkcije F. Točnije, $D(F) = \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$, a $\frac{\pi}{2} \in [0, \pi]$.

Primjer 16 *Pokažite da je* $F(x) = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{x+1}{\sqrt{2}}$ *primitivna funkcija funkcije* $f(x) = \frac{1}{x^2+2x+3}$ *te izračunajte:*

(a) $\int_0^1 \frac{dx}{x^2+2x+3}$

(b) $\int_0^1 f'(x)dx$

(c) $\int_0^1 F''(x)dx$

(d) $\left(\int_0^x f(t)dt\right)'$

(e) $\frac{d}{dx}\left(\int_0^1 f(x)dx\right)$

(f) $\frac{d}{dx}\left(\int_0^t f(x)dx\right)$

(g) $\frac{d}{dx}\left(\int_0^x f(t)dt\right)$ *za* $x = 2$.

Rješenje:
$F'(x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{1+\frac{(x+1)^2}{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\not{2}} \cdot \frac{1}{\frac{2+(x+1)^2}{\not{2}}} = \frac{1}{x^2+2x+3}$

(a) $\int_0^1 \frac{dx}{x^2+2x+3} = F(x)\Big|_0^1 = \frac{1}{\sqrt{2}} \operatorname{arctg} \sqrt{2} - \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{\sqrt{2}}{2}$

(b) $\int_0^1 f'(x)dx = f(x)\Big|_0^1 = \frac{1}{6} - \frac{1}{3} = -\frac{1}{6}$

(c)

$$\begin{aligned}\int_0^1 F''(x)dx &= \int_0^1 f'(x)dx = -\frac{1}{6}\\ &= F'(1) - F'(0) = -\frac{1}{6}\end{aligned}$$

(d) $\left(\int_0^x f(t)dt\right)' = \left(F(t)\Big|_0^x\right)' = (F(x)-F(0))' = F'(x) - 0 = f(x) = \frac{1}{x^2+2x+3}$

(e) $\frac{d}{dx}\left(\int_0^1 f(x)dx\right) = \frac{d}{dx}(F(1)-F(0)) = 0$

(f) $\frac{d}{dx}\left(\int_0^t f(x)dx\right) = \frac{d}{dx}(F(t)-F(0)) = 0$

(g) $\frac{d}{dx}\left(\int_0^x f(t)dt\right) = \frac{d}{dx}(F(x)-F(0)) = f(x)$ pa za $x=2$ imamo $f(2) = \frac{1}{11}$.

Primjer 17 *Izračunajte:*

(a) $\int_0^1 x^2 dx$

(b) $\int_a^b x^\alpha dx\,,\ \alpha \neq -1\,,\ a,b>0$

Rješenje:

(a) $\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$
Usporedite dobiveni rezultat s aproksimacijom u Primjeru 1 na str. 7.

(b) $\int_a^b x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}\Big|_a^b = \frac{b^{\alpha+1}-a^{\alpha+1}}{\alpha+1}$

Primjer 18 *(a) Izračunajte:* $\int_0^{2\pi} \sin x dx$

(b) Izračunajte površinu lika određenog s $y=\sin x$, $y=0$, $x=0$, $x=2\pi$

Rješenje:

(a) $\int_0^{2\pi} \sin x dx = -\cos x\Big|_0^{2\pi} = -1+1 = 0$

(b) Površinu danog lika dobiti ćemo integrirajući funkciju $|\sin x|$ u granicama od 0 do 2π tj.

$$\int_0^{2\pi} |\sin x| dx = \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx = -\cos x\Big|_0^{\pi} - \left(-\cos x\Big|_{\pi}^{2\pi}\right) = 4$$

Primjer 19 *Izračunajte:*

(a) $\int_1^3 \frac{dx}{x^2}$

(b) $\int_{-1}^1 \frac{dx}{x^2}$

RJEŠENJE:

(a) $\int_1^3 \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}$

(b) $\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^1 = -1 - 1 = -2$

Primijetimo da smo dobili rezultat koji nema smisla jer znamo da je $f(x) = \frac{1}{x^2} > 0$ pa bi i integral te funkcije na $[-1, 1]$ trebao biti veći od nule. Razlog zbog kojeg smo dobili apsurdan rezultat je taj što se unutar područja integracije nalazi točka koja nije u domeni podintegralne funkcije. Naime, $D(f) = \mathbb{R}\backslash\{0\}$, a $0 \in [-1, 1]$.

Primjer 20 *Izračunajte* $\int_{-\pi}^{\pi} (x^5 + 1) \cos^2 x dx$

RJEŠENJE:

$\int_{-\pi}^{\pi} (x^5 + 1) \cos^2 x dx = \int_{-\pi}^{\pi} x^5 \cos^2 x dx + \int_{-\pi}^{\pi} \cos^2 x dx =$

U prvom integralu imamo neparnu podintegralnu funkciju $x^5 \cos^2 x$ koju integriramo na području simetričnom oko nule pa znamo da je taj integral jednak nuli.

U drugom integralu imamo parnu podintegralnu funkciju $\cos^2 x$ (dakle, simetričnu s obzirom na y-os) koju integriramo na području simetričnom oko nule pa je dovoljno izračunati integral samo za pola tog područja i rezultat pomnožiti s dva.

$= 0 + 2 \int_0^{\pi} \cos^2 x dx = \int_0^{\pi} (1 + \cos 2x) dx = \pi$

Zadatak 46

$$\int_{-1}^{1} (2x^2 - x^3) dx = \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_{-1}^{1} = \left(\frac{2}{3} - \frac{1}{4}\right) - \left(-\frac{2}{3} - \frac{1}{4}\right) = \frac{4}{3}$$

Zadatak 47

$$\int_{-2}^{2} \frac{dx}{x^2+4} = \frac{1}{2}\operatorname{arctg}\frac{x}{2}\Big|_{-2}^{2} = \frac{1}{2}\operatorname{arctg}1 - \frac{1}{2}\operatorname{arctg}(-1) = \frac{1}{2}\frac{\pi}{4} - \frac{1}{2}\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

Zadatak 48

$$\int_{\pi/2}^{3\pi/4} \sin x dx = -\cos x\Big|_{\pi/2}^{3\pi/4} = -\cos\frac{3\pi}{4} + \cos\frac{\pi}{2} = \frac{\sqrt{2}}{2} + 0 = \frac{\sqrt{2}}{2}$$

Zadatak 49

$$\int_{2}^{5} \sqrt{x+3}\, dx = \int_{2}^{5} \sqrt{x+3}\, d(x+3) = \frac{(x+3)^{3/2}}{\frac{3}{2}}\Big|_{2}^{5} = \frac{2}{3}(8\sqrt{8} - 5\sqrt{5}) = \frac{2}{3}(16\sqrt{2} - 5\sqrt{5})$$

Zadatak 50

$$\int_{0}^{1} \frac{dx}{x^2+2x+4} = \int_{0}^{1} \frac{dx}{(x+1)^2+3} = \frac{1}{3}\int_{0}^{1} \frac{dx}{(\frac{x+1}{\sqrt{3}})^2+1} = \frac{1}{\sqrt{3}}\int_{0}^{1} \frac{d(\frac{x+1}{\sqrt{3}})}{(\frac{x+1}{\sqrt{3}})^2+1}$$

$$= \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+1}{\sqrt{3}}\Big|_{0}^{1} = \frac{1}{\sqrt{3}}\left(\operatorname{arctg}\frac{2}{\sqrt{3}} - \operatorname{arctg}\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(\operatorname{arctg}\frac{2}{\sqrt{3}} - \frac{\pi}{6}\right)$$

Zadatak 51

$$\int_{-2}^{2} \frac{2x+1}{x+3}\, dx = \int_{-2}^{2} \frac{2(x+3)-5}{x+3}\, dx = 2\int_{-2}^{2} dx - 5\int_{-2}^{2} \frac{1}{x+3}\, dx$$

$$= 2x\Big|_{-2}^{2} - 5\ln|x+3|\Big|_{-2}^{2} = 2(2-(-2)) - 5(\ln 5 - \ln 1) = 8 - 5\ln 5$$

Zadatak 52

$$\int_{0}^{100\pi} \sqrt{1-\cos 2x}dx = \int_{0}^{100\pi} \sqrt{2\sin^2 x}dx = 100\sqrt{2}\int_{0}^{\pi} |\sin x|dx$$

$$= \quad 100\sqrt{2}\int_{0}^{\pi} \sin x dx = -100\sqrt{2}\cos x\Big|_{0}^{\pi} = 200\sqrt{2}$$

Zadatak 53 *Neka je* $F(x) = \int_{\frac{1}{x}}^{x^2} \frac{dt}{t+2}$*. Odredite: a)* $F(1)$ *b)* $F(2)$ *c)* $F'(x)$

Zadatak 54 *Neka je* $F(x) = \int_{\sqrt{x}}^{x^2} \sqrt{1+t^2}dt$*. Odredite: a)* $D(F)$ *b)* $N(f)$ *c)* $x \in D(F)$ *za koje je* $F(x) > 0$ *i* $F(x) < 0$ *d)* $F'(x)$

Zadatak 55 *Izračunajte:*

a) $\frac{d}{dx}\left(\int_0^{1+x^2} \frac{dt}{\sqrt{2t+5}}\right)$

b) $\frac{d}{dx}\left(\int_{x^2}^{3} \frac{\sin t}{t}\, dt\right)$

c) $\frac{d}{dx}\left(\int_{3x}^{\frac{1}{x}} \cos(2t)\, dt\right)$

d) $\frac{d}{dx}\left(\int_{1}^{3} \frac{\sin x}{x}\, dx\right)$

e) $\frac{d}{dx}\left(\int_{0}^{x} (x^2+t)\, dt\right)$

f) $\frac{d}{dx}\left(\int_{0}^{t} x\sin x\, dx\right)$

Zadatak 56 *Izračunajte:* $\lim\limits_{x\to 0} \frac{\int_0^x \sqrt{1+t^4}\, dt}{x}$.

Zadatak 57 *Odredite stacionarne točke funkcije* $F(x) = \int_0^{x^2} \frac{t-1}{\sqrt{1+t^2}}\, dt$.

Zadatak 58 *Odredite* $F'(2)$ *ako je* $F(x) = \int_{2x}^{x^3-4} \frac{x}{1+\sqrt{t}}\, dt$.

3.1 Supstitucija u određenom integralu

Jednostavna posljedica Newton-Leibnizove formule i formule za derivaciju kompozicije funkcija je sljedeći teorem koji daje uvjete za supstituciju u određenom integralu.

Teorem 4 *Neka je* $f : [a,b] \to \mathbb{R}$ *neprekidna funkcija i neka je* $\varphi : [\alpha, \beta] \to$ **R** *funkcija sa neprekidnom prvom derivacijom tako da je* $\varphi(\alpha) = a$ *i* $\varphi(\beta) = b$. *Tada je*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

Zadatak 59

$$\int_0^{\frac{2\pi}{3}} \frac{dx}{5+4\cos x} = \left\{ \begin{array}{c} t=\operatorname{tg}\frac{x}{2} \quad t(\frac{2\pi}{3})=\sqrt{3} \quad t(0)=0 \\ dx=\frac{2dt}{1+t^2} \Rightarrow \cos x = \frac{1-t^2}{1+t^2} \end{array} \right\} =$$

$$\int_0^{\sqrt{3}} \frac{\frac{2dt}{1+t^2}}{5+4\frac{1-t^2}{1+t^2}} = 2\int_0^{\sqrt{3}} \frac{dt}{9+t^2} = 2\cdot\frac{1}{3}\cdot \operatorname{arctg}\frac{t}{3}\Big|_0^{\sqrt{3}} = \frac{2}{3}\cdot\frac{\pi}{6} = \frac{\pi}{9}$$

Zadatak 60

$$\int_0^{100\pi} \frac{dx}{4+\cos x} = 50\cdot\int_0^{2\pi} \frac{dx}{4+\cos x} = \left\{ \begin{array}{c} x' = x-\pi \\ x = x'+\pi \end{array} \right\} = 50\cdot\int_{-\pi}^{\pi} \frac{dx'}{4-\cos x'}$$

$$= 100\int_0^{\pi} \frac{dx'}{4-\cos x'} = \left\{ \begin{array}{c} t=\operatorname{tg}\frac{x'}{2} \in \langle 0,\infty\rangle \\ dx' = \frac{2dt}{1+t^2} \end{array} \right\} = 100\cdot\int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{4-\frac{1-t^2}{1+t^2}}$$

$$= 100\cdot\int_0^{\infty} \frac{2dt}{4+4t^2-1+t^2} = 100\int_0^{\infty}\frac{2dt}{3+5t^2} = 40\int_0^{\infty}\frac{dt}{\frac{3}{5}+t^2}$$

$$= 40\cdot\sqrt{\frac{5}{3}}\cdot\operatorname{arctg}\frac{t\sqrt{5}}{\sqrt{3}}\Big|_0^{\infty} = 40\frac{\sqrt{15}}{3}\cdot\frac{\pi}{2} = \frac{20}{3}\sqrt{15}\pi$$

Zadatak 61

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}dx = \left\{ \begin{array}{c} x=\frac{1}{\cos t} \\ \\ dx = \frac{\sin t\, dt}{\cos^2 t} \end{array} \right\} = \int_0^{\frac{\pi}{3}} \frac{\sqrt{\frac{1}{\cos^2 t}-1}}{\frac{1}{\cos t}}\frac{\sin t\, dt}{\cos^2 t} = \int_0^{\frac{\pi}{3}}\frac{\sin^2 t}{\cos^2 t}dt$$

$$= \int_0^{\frac{\pi}{3}} \frac{1-\cos^2 t}{\cos^2 t}dt = \int_0^{\frac{\pi}{3}}\frac{1}{\cos^2 t}dt - \int_0^{\frac{\pi}{3}} dt = \operatorname{tg} t\Big|_0^{\frac{\pi}{3}} - t\Big|_0^{\frac{\pi}{3}} = \sqrt{3}-\frac{\pi}{3}$$

2.način

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}dx = \left\{ \begin{array}{ll} x=\frac{1}{t} & x=1 \Rightarrow t=1 \\ dx=-\frac{dt}{t^2} & x=2 \Rightarrow t=1/2 \end{array} \right\} = -\int_1^{1/2} \frac{\sqrt{\frac{1}{t^2}-1}}{\frac{1}{t}}\frac{dt}{t^2}$$

$$= \int_{1/2}^1 \frac{\sqrt{1-t^2}}{t^2}\,dt = \left\{ \begin{array}{ll} t=\sin y & t=1/2 \Rightarrow y=\pi/6 \\ dt=\cos y dy & t=1 \Rightarrow y=\pi/2 \end{array} \right\}$$

$$= \int_{\pi/6}^{\pi/2} \frac{\sqrt{1-\sin^2 y}}{\sin^2 y}\,\cos y\, dy = \int_{\pi/6}^{\pi/2}\frac{\cos^2 y}{\sin^2 y}\,dy = \int_{\pi/6}^{\pi/2}\frac{1-\sin^2 y}{\sin^2 y}\,dy$$

$$= \int_{\pi/6}^{\pi/2}\frac{dy}{\sin^2 y} - \int_{\pi/6}^{\pi/2} dy = -\operatorname{ctg} y\Big|_{\pi/6}^{\pi/2} - y\Big|_{\pi/6}^{\pi/2} = -\operatorname{ctg}\frac{\pi}{2} + \operatorname{ctg}\frac{\pi}{6} - \frac{\pi}{2} + \frac{\pi}{6} = \sqrt{3}-\frac{\pi}{3}$$

Zadatak 62

$$\int_4^9 \frac{1-\sqrt{x}}{1+\sqrt{x}}dx = \left\{ \begin{array}{cc} t^2 = x & t(9) = 3 \\ 2tdt = dx & t(4) = 2 \end{array} \right\} = \int_2^3 \frac{1-t}{1+t} \cdot 2tdt = 2\int_2^3 \frac{t-t^2}{1+t}dt$$

$$= \left((-t^2+t) : (t+1) = -t + 2 - \frac{2}{t+1} \right)$$

$$= -2 \cdot \int_2^3 tdt + 4 \cdot \int_2^3 dt - 4 \cdot \int_2^3 \frac{dt}{1+t} = -2 \cdot \frac{t^2}{2}\Big|_2^3 + 4t\Big|_2^3 - 4\ln|1+t|\Big|_2^3 =$$

$$= -9 + 4 + 12 - 8 - 4\ln|4| + 4\ln|3| = -1 + 4\ln\frac{3}{4}$$

3.2 Parcijalna integracija u određenom integralu

Kako je $f(x)g(x)$ primitivna funkcija funkcije $f(x)g'(x)+f'(x)g(x)$ to korištenjem Newton-Leibnizove formule slijedi sljedeći teorem.

Teorem 5 *Neka su $f, g : [a, b] \to \mathbb{R}$ funkcije sa neprekidnom prvom derivacijom. Tada je*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)dx.$$

Prethodni teorem također možemo zapisati u diferencijalnom obliku:

$$\int_a^b udv = u \cdot v\Big|_a^b - \int_a^b vdu$$

Zadatak 63

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{xdx}{\sin^2 x} = \left\{ \begin{array}{cc} u = x & dv = \frac{dx}{\sin^2 x} \\ du = dx & v = -\operatorname{ctg} x \end{array} \right\} = -x\operatorname{ctg} x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sin x}dx =$$

$$= \left\{ \begin{array}{cc} t = \sin x & t(\frac{\pi}{6}) = \frac{1}{2} \\ dt = \cos xdx & t(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \end{array} \right\} = -\frac{\pi}{4} + \frac{\pi}{6}\sqrt{3} + \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{t}dt = -\frac{\pi}{4} + \frac{\sqrt{3}\pi}{6} + \ln|t|\Big|_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} =$$

$$= \frac{2\sqrt{3}\pi - 3\pi}{12} + \ln\sqrt{2}$$

Zadatak 64

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^{10}+x)\sin x dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^{10}\sin x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin x dx = 2\int_0^{\pi/2} x\sin x\ dx$$

$$= \left\{ \begin{array}{cc} u = x & dv = \sin x dx \\ du = dx & v = -\cos x \end{array} \right\} =$$

$$= 2\cdot\left(-x\cos x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x dx \right) = 2\cdot\left(0 + \sin x\Big|_0^{\frac{\pi}{2}} \right) = 2$$

Zadatak 65 *(DZ)*

$$\int_0^1 \arcsin x\ dx = \textit{(vidi Zad.19b)} = \ldots = \frac{\pi}{2} - 1$$

Zadatak 66 *(DZ)*

$$\int_1^{e^2} \ln x\ dx = \ldots = 1 + e^2$$

Zadatak 67 *(DZ)*

$$a) \qquad \int_0^2 \sqrt{1+x^2}\ dx = \textit{(vidi Zad.20b)} = \ldots = \sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})$$

$$b) \qquad \int_1^3 \sqrt{x^2-1}\ dx = \textit{(vidi Zad.20a)} = \ldots = 3\sqrt{2} - \frac{1}{2}\ln(3+2\sqrt{2})$$

4 Primjena određenog integrala

4.1 Kvadratura (površina ravninskih likova)

Kartezijeve koordinate

Ako je krivocrtni trapez u ravnini zadan sa

$$D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}$$

onda je površina od D dana sa

$$P = \int_a^b (f_2(x) - f_1(x))dx.$$

Parametarski oblik: Ako je krivulja $y = f(x)$ zadana parametarski sa $x = \varphi(t),\ y = \psi(t)$ (φ je strogo monotona funkcija) onda je površina od

$$D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, 0 \leq y \leq f(x)\}$$

dana sa

$$P = \int_{t_1}^{t_2} \psi(t)\varphi'(t)dt.$$

Polarne koordinate

Ako je krivocrtni isječak u ravnini zadan u polarnim koordinatama sa

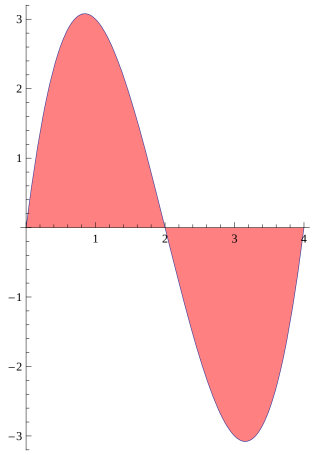
$$D = \{(\varphi, r) \in \mathbb{R} \times [0, \infty\rangle; a \leq \varphi \leq \beta, 0 \leq r_1(\varphi) \leq r \leq r_2(\varphi)\}$$

onda je površina od D dana sa

$$P = \frac{1}{2}\int_\alpha^\beta (r_2(\varphi)^2 - r_1(\varphi)^2)d\varphi.$$

Zadatak 68 *Izračunajte površinu lika omeđenog sa* $y = x^3 - 6x^2 + 8x,\ y = 0.$

Rješenje:

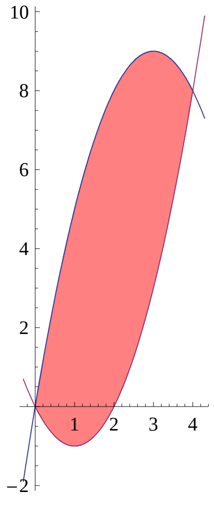


$$\begin{aligned} P &= \int_0^2 (x^3 - 6x^2 + 8x)dx - \int_2^4 (x^3 - 6x^2 + 8x)dx \\ &= \left(\frac{x^4}{4} - 2x^3 + 4x^2\right)\Big|_0^2 + \left(-\frac{x^4}{4} + 2x^3 - 4x^2\right)\Big|_2^4 = 8. \end{aligned}$$

□

Zadatak 69 *Izračunajte površinu lika omeđenog sa* $y = 6x - x^2,\ y = x^2 - 2x.$

Rješenje:



$$P = \int_0^4 [6x - x^2 - (x^2 - 2x)]dx = \int_0^4 (8x - 2x^2)dx = \left(4x^2 - \frac{2}{3}x^3\right)\Big|_0^4 = \frac{64}{3}.$$

□

Zadatak 70 *Izračunajte površinu lika omeđenog sa* $y = x(x+3)(x-2)$, $y = x(2-x)$.

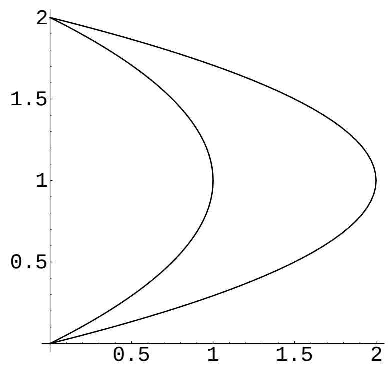
Zadatak 71 *Izračunajte površinu lika omeđenog sa:*

a) $y^2 = x,\ y^2 = 4x,\ y = x$

b) $y = 2x,\ y = 4x,\ y^2 = 2x$

Zadatak 72 *Izračunajte površinu lika omeđenog sa* $2(y-1)^2 = 2-x$, $(y-1)^2 = 1-x$.

Rješenje:

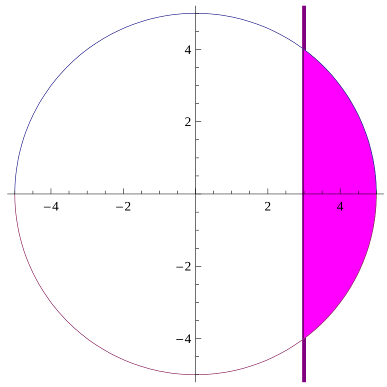


$$P = \int_0^2 [2-2(y-1)^2-(1-(y-1)^2)]dy = \int_0^2 (-y^2+2y)dy = -\frac{y^3}{3}\Big|_0^2 + y^2\Big|_0^2 = \frac{4}{3}.$$

□

Zadatak 73 *Izračunajte površinu lika omeđenog sa* $x^2 + y^2 \leq 25,\ x \geq 3$ *a) u kartezijevim koordinatama b) u polarnim koordinatama.*

Rješenje: a)

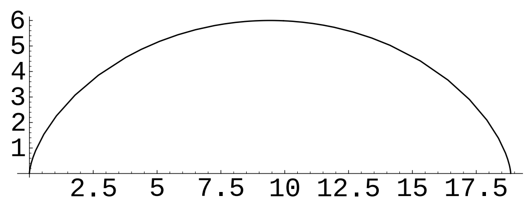


$$
\begin{aligned}
P &= 2\int_3^5 \sqrt{25-x^2}dx = \left\{ \begin{array}{c} x = 5\sin t,\ t = \arcsin\frac{x}{5} \\ dx = 5\cos t dt \end{array} \right\} \\
&= 50\int_{\arcsin(3/5)}^{\pi/2} \cos^2 t dt = 25\int_{\arcsin(3/5)}^{\pi/2} (1+\cos 2t)dt \\
&= 25t\Big|_{\arcsin(3/5)}^{\pi/2} + 25\sin t\sqrt{1-\sin^2 t}\Big|_{\arcsin(3/5)}^{\pi/2} = \frac{25\pi}{2} - 25\arcsin\frac{3}{5} - 12.
\end{aligned}
$$

□

Zadatak 74 *Izračunajte površinu lika omeđenog prvim svodom cikloide* $x = 3(t-\sin t),\ y = 3(1-\cos t)$ *i osi apscise.*

Rješenje:

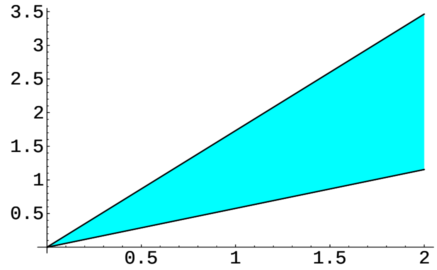


$$
\begin{aligned}
P &= \int_0^{2\pi} 3(1-\cos t)\cdot 3(1-\cos t)dt = 9\int_0^{2\pi}(1-2\cos t+\cos^2 t)dt \\
&= 9t\Big|_0^{2\pi} - 18\sin t\Big|_0^{2\pi} + \frac{9}{2}\int_0^{2\pi}(1+\cos 2t)dt = 18\pi + \frac{9}{2}t\Big|_0^{2\pi} + \frac{9}{4}\sin 2t\Big|_0^{2\pi} = 27\pi.
\end{aligned}
$$

□

Zadatak 75 *Izračunajte površinu lika omeđenog sa* $r = \frac{2}{\cos\varphi}$, $\varphi = \frac{\pi}{6}$, $\varphi = \frac{\pi}{3}$.

Rješenje:

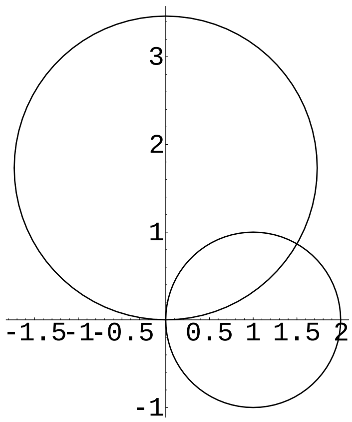


$$P = \frac{1}{2}\int_{\pi/6}^{\pi/3} \frac{4}{\cos^2\varphi} d\varphi = 2\operatorname{tg}\varphi \Big|_{\pi/6}^{\pi/3} = \frac{4\sqrt{3}}{3}.$$

□

Zadatak 76 *Izračunajte površinu lika omeđenog sa* $x^2 + y^2 \leq 2\sqrt{3}y$, $x^2 + y^2 \leq 2x$.

Rješenje:

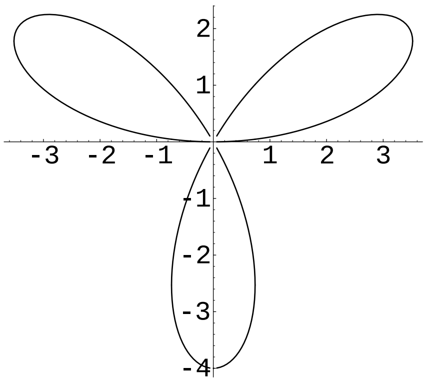


$$\begin{aligned}
P &= \frac{1}{2}\int_0^{\pi/6} 12\sin^2\varphi d\varphi + \frac{1}{2}\int_{\pi/6}^{\pi/2} 4\cos^2\varphi d\varphi \\
&= 3\int_0^{\pi/6} (1-\cos 2\varphi) d\varphi + \int_{\pi/6}^{\pi/2} (1+\cos 2\varphi) d\varphi \\
&= 3\varphi\Big|_0^{\pi/6} - \frac{3\sin 2\varphi}{2}\Big|_0^{\pi/6} + \varphi\Big|_{\pi/6}^{\pi/2} + \frac{\sin 2\varphi}{2}\Big|_{\pi/6}^{\pi/2} = \frac{5\pi}{6} - \sqrt{3}.
\end{aligned}$$

□

Zadatak 77 *Izračunajte površinu lika omeđenog sa $r = 4\sin 3\varphi$.*

Rješenje:



$$P = \frac{3}{2}\int_0^{\pi/3}(4\sin 3\varphi)^2 d\varphi = 24\int_0^{\pi/3}\frac{1-\cos 6\varphi}{2}d\varphi = 12\varphi\Big|_0^{\pi/3} - 2\sin 6\varphi\Big|_0^{\pi/3} = 4\pi.$$

□

4.2 Rektifikacija (duljina luka krivulje)

Kartezijeve koordinate

Ako je krivulja zadana sa $y = f(x), x \in [a, b]$ onda je njezina duljina dana sa

$$s = \int_a^b \sqrt{1 + y'(x)^2}dx.$$

Parametarski oblik: Ako je krivulja zadana parametarski sa $x = \varphi(t), y = \psi(t)$ onda je njezina duljina dana sa

$$s = \int_{t_1}^{t_2} \sqrt{\varphi'(t)^2 + \psi'(t)^2}dt.$$

Polarne koordinate

Ako je krivulja zadana sa $r = r(\varphi), \varphi \in [\alpha, \beta]$ onda je njezina krivulja dana sa

$$s = \int_\alpha^\beta \sqrt{r(\varphi)^2 + r'(\varphi)^2}d\varphi.$$

Zadatak 78 *Izračunajte duljinu luka krivulje* $y = x^{3/2}$ *od* $x = 0$ *do* $x = 5$.

Rješenje:

$$s = \int_0^5 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}dx = \frac{1}{2}\int_0^5 \sqrt{4+9x}dx = \frac{1}{18}\cdot\frac{2}{3}(4+9x)\sqrt{4+9x}\Big|_0^5 = \frac{335}{27}.$$

□

Zadatak 79 *Izračunajte duljinu luka krivulje* $y = x^{2/3}$ *od* $x = 0$ *do* $x = 8$.

Rješenje:

$$s = \int_0^4 \sqrt{1 + \left(\frac{3}{2}y^{1/2}\right)^2}dy = \frac{1}{2}\int_0^4 \sqrt{4+9y}dy = \frac{1}{18}\cdot\frac{2}{3}(4+9y)\sqrt{4+9y}\Big|_0^4 = \frac{8}{27}(10\sqrt{10}-1).$$

□

Zadatak 80 *Koji put prevali čestica koja se kreće po krivulji* $x = \cos^2 t,\ y = \sin^2 t$ *u vremenu od* $t = 0$ *do* $t = 2\pi$.

Rješenje:

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{4\cos^2 t \sin^2 t + 4\sin^2 t\cos^2 t}dt = 2\sqrt{2}\int_0^{2\pi} |\cos t \sin t| dt \\ &= 2\sqrt{2}\int_0^{\pi} |\sin 2t| dt = 2\sqrt{2}\int_0^{\pi/2} \sin 2t dt - 2\sqrt{2}\int_{\pi/2}^{\pi} \sin 2t dt \\ &= -\sqrt{2}\cos 2t\Big|_0^{\pi/2} + \sqrt{2}\cos 2t\Big|_{\pi/2}^{\pi} = 4\sqrt{2}. \end{aligned}$$

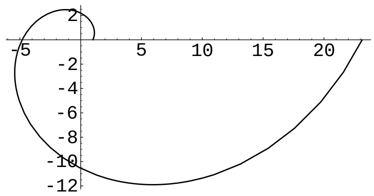
□

Zadatak 81 *Izračunajte duljinu prvog luka logaritamske zavojnice* $r = e^{\frac{1}{2}\varphi}$ *od* $\varphi = 0$ *do* $\varphi = 2\pi$.

Rješenje:

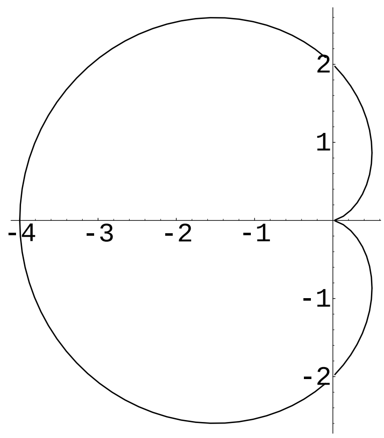
$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{e^{\varphi} + \left(\frac{1}{2}e^{\frac{1}{2}\varphi}\right)^2}d\varphi = \frac{\sqrt{5}}{2}\int_0^{2\pi} \sqrt{e^{\varphi}}d\varphi = \frac{\sqrt{5}}{2}\int_0^{2\pi} e^{\frac{1}{2}\varphi}d\varphi \\ &= \sqrt{5}e^{\frac{1}{2}\varphi}\Big|_0^{2\pi} = \sqrt{5}e^{\pi} - \sqrt{5}. \end{aligned}$$

□



Zadatak 82 *Izračunajte opseg kardioide* $r = 2(1 - \cos\varphi)$.

Rješenje:



$$
\begin{aligned}
s &= \int_0^{2\pi} \sqrt{4(1-\cos\varphi)^2 + 4\sin^2\varphi}\, d\varphi = 2\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\varphi}\, d\varphi \\
&= 4\int_0^{2\pi}\left|\sin\frac{\varphi}{2}\right| d\varphi = 4\int_0^{2\pi}\sin\frac{\varphi}{2} d\varphi = -8\cos\frac{\varphi}{2}\Big|_0^{2\pi} = 16.
\end{aligned}
$$

□

4.3 Kubatura (volumen tijela)

4.3.1 Kubatura rotacijskih tijela

Rotacija oko osi paralelnih sa osi apscisa:

Ako područje

$$D = \{(x, y) : a \le x \le b,\ y_0 \le y \le f(x) \text{ ili } f(x) \le y \le y_0\}$$

(područje D je omeđeno sa krivuljom $y = f(x)$ i pravcima $x = a$, $x = b$, $y = y_0$) rotira oko pravca $y = y_0$ dobije se tijelo volumena

$$V_{y=y_0} = \pi \int_a^b [f(x) - y_0]^2 \, dx = \pi \int_a^b [y - y_0]^2 \, dx.$$

Specijalno ako je $y_0 = 0$ (rotacija oko $x-$osi) formula glasi

$$V_{y=0} = \pi \int_a^b f^2(x) dx = \pi \int_a^b y^2 dx.$$

Ako područje

$$D = \{(x, y) : a \leq x \leq b, 0 \leq f(x) \leq y \leq g(x) \text{ ili } g(x) \leq y \leq f(x) \leq 0\}$$

(područje D je omeđeno krivuljama $y = f(x)$, $y = g(x)$ i pravcima $x = a$, $x = b$ i cijelo se nalazi ili "ispod" ili "iznad" osi apscisa pri čemu je krivulja $y = g(x)$ "udaljenija" od osi apscisa) rotira oko osi apscisa dobije se tijelo volumena

$$V_{y=0} = \pi \int_a^b [g^2(x) - f^2(x)] dx.$$

Rotacija oko osi paralelne sa osi ordinata:

Ako je područje D omeđeno krivuljom $y = f(x)$, pravcima $x = a$, $x = b$ i $y = 0$ i cijelo se nalazi ili sa "desne" ili sa "lijeve" strane pravca $x = x_0$, onda njegovom rotacijom oko pravca $x = x_0$ nastaje tijelo volumena

$$V_{x=x_0} = 2\pi \int_a^b |x - x_0| \, |f(x)| dx = 2\pi \int_a^b |x - x_0| \, |y| dx.$$

Specijalno, ako je $x_0 = 0$ (rotacija oko $y-$ osi) formula glasi

$$V_{x=0} = 2\pi \int_a^b |x||y| dx.$$

Najjednostavniji slučaj je kada se cijelo područje nalazi "iznad" osi apscisa ($y \geq 0$) i "desno" od osi ordinata ($x \geq 0$). U tom slučaju volumen tijela nastalog rotacijom područja

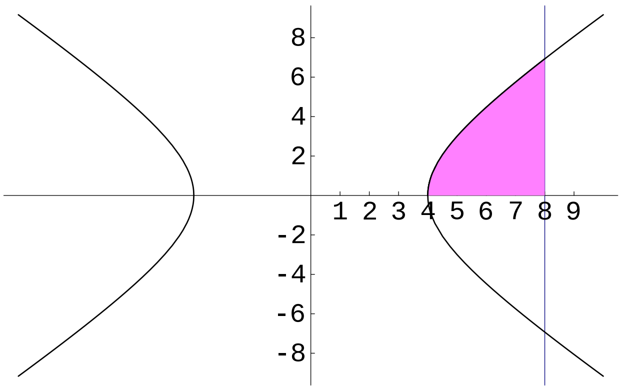
$$D = \{(x, y) : 0 \leq a \leq x \leq b, \; 0 \leq y \leq f(x)\}$$

oko osi ordinata glasi

$$V_{x=0} = 2\pi \int_a^b x f(x) dx.$$

Zadatak 83 *Površina omeđena sa* $x^2 - y^2 = 16,\ y = 0,\ x = 8$ *rotira oko* x*-osi. Odredite volumen nastalog rotacionog tijela.*

Rješenje:

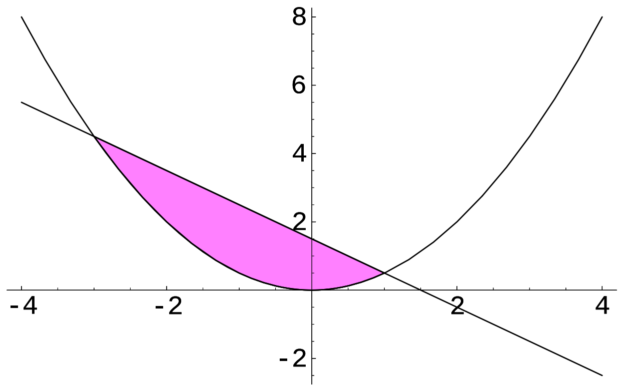


$$V_{y=0} = \pi \int_4^8 (x^2 - 16) dx = \pi \left[\frac{x^3}{3} \Big|_4^8 - 16x \Big|_4^8 \right] = \frac{256\pi}{3}.$$

□

Zadatak 84 *Površina omeđena sa* $y = \frac{x^2}{2}$, $y = \frac{3}{2} - x$ *rotira oko* x*-osi. Odredite volumen nastalog rotacionog tijela.*

Rješenje:



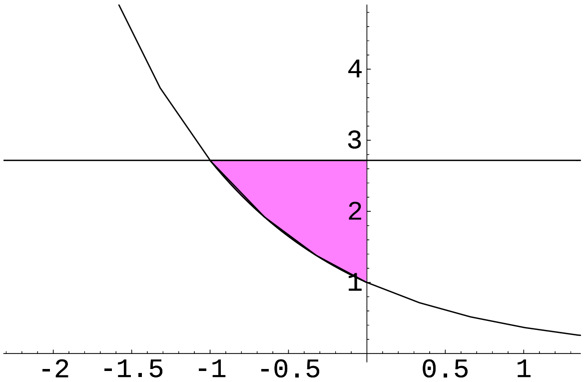
$$\begin{aligned} V_{y=0} &= \pi \int_{-3}^{1} \left[\left(\frac{3}{2} - x \right)^2 - \left(\frac{x^2}{2} \right)^2 \right] dx = \pi \int_{-3}^{1} \left(\frac{9}{4} - 3x + x^2 - \frac{x^4}{4} \right) dx \\ &= \pi \left[\frac{9x}{4} - \frac{3x^2}{2} + \frac{x^3}{3} - \frac{x^5}{20} \right] \Bigg|_{-3}^{1} = \frac{89\pi}{15}. \end{aligned}$$

$\square$

Zadatak 85 *Površina omeđena sa* $y = e^{-x}$, $y = e$, $x = 0$ *rotira oko* y*-osi. Odredite volumen nastalog rotacionog tijela.*

Rješenje: Prvi način:

$$\begin{aligned} V_{x=0} &= \pi \int_{1}^{e} (-\ln y)^2 dy = \pi \int_{1}^{e} \ln^2 y dy = \left\{ \begin{array}{c} u = \ln^2 y \Rightarrow du = \frac{2 \ln y}{y} dy \\ dv = dy \Rightarrow v = y \end{array} \right\} \\ &= \pi y \ln^2 y \Big|_{1}^{e} - 2\pi \int_{1}^{e} \ln y dy = \left\{ \begin{array}{c} u = \ln y \Rightarrow du = \frac{1}{y} dy \\ dv = dy \Rightarrow v = y \end{array} \right\} \\ &= e\pi - 2\pi y \ln y \Big|_{1}^{e} + 2\pi \int_{1}^{e} dy = e\pi - 2e\pi + 2\pi y \Big|_{1}^{e} = \pi(e-2). \end{aligned}$$



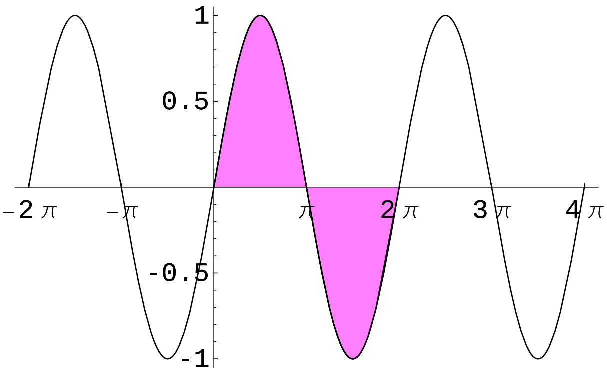
Drugi način:

$$
\begin{aligned}
V_{x=0} &= 2\pi \int_{-1}^{0} |x|(e - e^{-x})dx = 2\pi \left[\int_{-1}^{0} xe^{-x}dx - e\int_{-1}^{0} xdx \right] \\
&= \left\{ \begin{array}{c} u = x \Rightarrow du = dx \\ dv = e^{-x}dx \Rightarrow v = -e^{-x} \end{array} \right\} = 2\pi \left[-xe^{-x}\Big|_{-1}^{0} + \int_{-1}^{0} e^{-x}dx - e\frac{x^2}{2}\Big|_{-1}^{0} \right] \\
&= 2\pi \left[-e - e^{-x}\Big|_{-1}^{0} + \frac{e}{2} \right] = \pi(e-2).
\end{aligned}
$$

□

Zadatak 86 *Površina omeđena sa* $y = \sin x,\ y = 0,\ 0 \le x \le 2\pi$ *rotira oko* y*-osi. Odredite volumen nastalog rotacionog tijela.*

Rješenje:

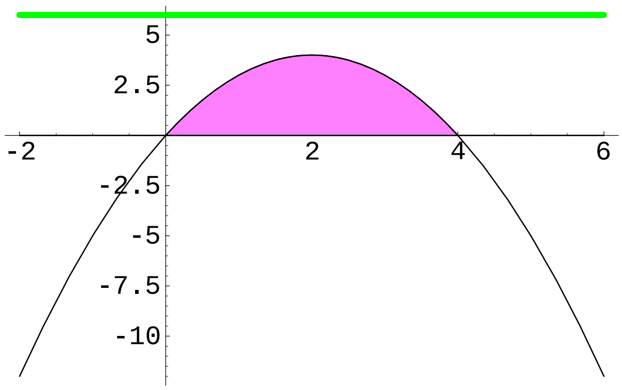


$$
\begin{aligned}
V_{x=0} &= 2\pi\left[\int_0^{\pi} x\sin x dx + \int_{\pi}^{2\pi} x(-\sin x)dx\right] = \left\{\begin{array}{c} u = x \Rightarrow du = dx \\ dv = \sin x dx \Rightarrow v = -\cos x \end{array}\right\} \\
&= 2\pi\left[-x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x dx + x\cos x\Big|_{\pi}^{2\pi} - \int_{\pi}^{2\pi}\cos x dx\right] = 8\pi^2.
\end{aligned}
$$

□

Zadatak 87 *Površina omeđena sa $y = 4x - x^2$, $y = 0$ rotira oko pravca a) $y = 6$, b) $y = -6$. Odredite volumen nastalog rotacionog tijela.*

Rješenje: a)

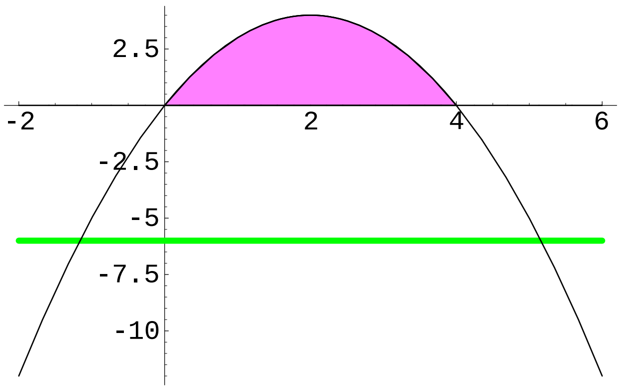


$$
\begin{aligned}
V_{y=6} &= \pi\int_0^4 [(-6)^2 - (4x - x^2 - 6)^2]dx = \pi\int_0^4 (-x^4 + 8x^3 - 28x^2 + 48x)dx \\
&= \pi\left[-\frac{x^5}{5} + 2x^4 - \frac{28x^3}{3} + 24x^2\right]\Big|_0^4 = \frac{1408}{15}\pi
\end{aligned}
$$

b)

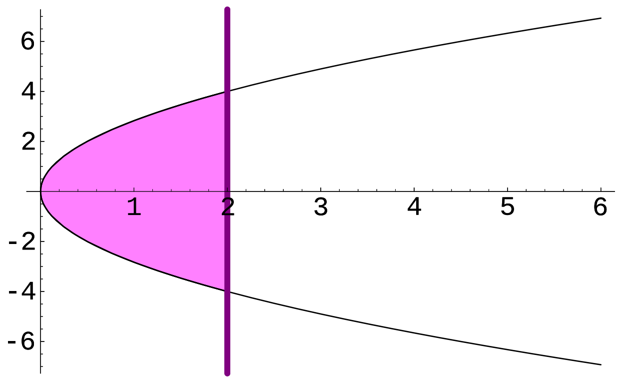
$$
V_{y=-6} = \pi\int_0^4 [(4x - x^2 + 6)^2 - 6^2]dx = \ldots = \frac{2432}{15}\pi.
$$

□



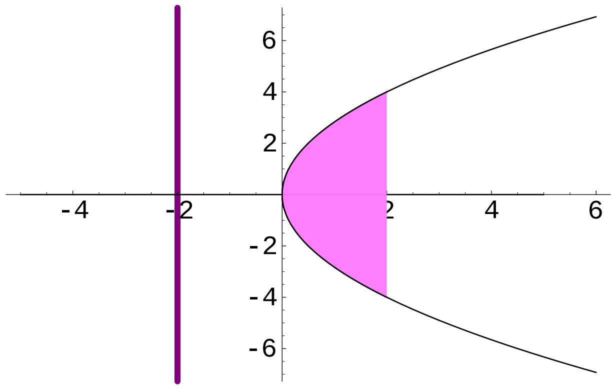
Zadatak 88 *Površina omeđena sa* $y^2 = 8x$, $x = 2$ *rotira oko pravca a)* $x = 2$, *b)* $x = -2$. *Odredite volumen nastalog rotacionog tijela.*

Rješenje: a)



$$V_{x=2} = \pi \int_{-4}^{4} \left(\frac{y^2}{8} - 2 \right)^2 dy = \frac{\pi}{64} \left[\frac{y^5}{5} - 32\frac{y^3}{3} + 256y \right] \Big|_{-4}^{4} = \frac{256}{15}\pi.$$

b)

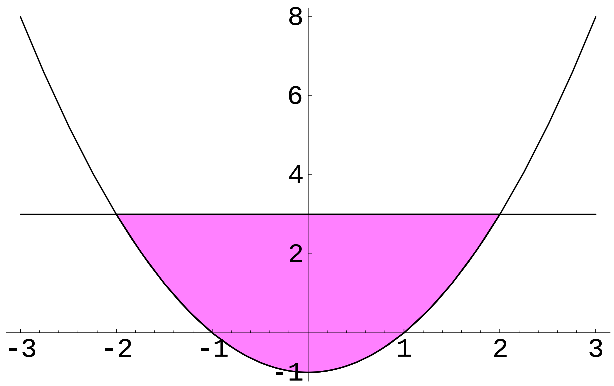


$$V_{x=-2} = \pi \int_{-4}^{4} \left[4^2 - \left(\frac{y^2}{8} + 2 \right)^2 \right] dy = \ldots = \frac{1024}{15}\pi.$$

□

Zadatak 89 *Površina omeđena sa* $y = x^2 - 1$, $y = 3$ *rotira oko* x*-osi. Odredite volumen nastalog rotacionog tijela.*

Rješenje:



$$\begin{aligned} V_{y=0} &= 2\pi\int_0^2 9dx - 2\pi\int_1^2 (x^2-1)^2 dx = 18\pi x\Big|_0^2 - 2\pi\int_1^2 (x^4-2x^2+1)dx \\ &= 36\pi - 2\pi\left[\frac{x^5}{5} - \frac{3x^3}{3} + x\right]\Bigg|_1^2 = \frac{464\pi}{15} \end{aligned}$$

$\square$